ECE 302: Lecture A.4 Auto-covariance, and Independent Processes

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Autocovariance Function

The **autocovariance function** of a random process $X(t)$ is

$$C_X(t_1, t_2) = \mathbb{E}\left[(X(t_1) - \mu_X(t_1))\,(X(t_2) - \mu_X(t_2))\right].$$

Two useful properties:

1. $C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$
2. $C_X(t, t) = \mathrm{Var}\left[X(t)\right]$

Proof.

Example

Example 1. Suppose $X(t) = A\cos(2\pi t)$ for $A \sim \text{Uniform}[0,1]$. Find $C_X(t_1, t_2)$.

Solution.

$$
\begin{aligned}
C_X(t_1, t_2) &= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) \\
&= \frac{1}{3}\cos(2\pi t_1)\cos(2\pi t_2) - \frac{1}{2}\cos(2\pi t_1) \cdot \frac{1}{2}\cos(2\pi t_2) \\
&= \frac{1}{12}\cos(2\pi t_1)\cos(2\pi t_2).
\end{aligned}
$$

Example

Example 2. Suppose $X(t) = \cos(\omega t + \Theta)$ for $\Theta \sim \text{Uniform}[-\pi, \pi]$. Find $C_X(t_1, t_2)$.

Solution.

$$
\begin{aligned}
C_X(t_1, t_2) &= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) \\
&= \frac{1}{2}\cos\Big(\omega(t_1 - t_2)\Big) - 0 \cdot 0 = \frac{1}{2}\cos\Big(\omega(t_1 - t_2)\Big).
\end{aligned}
$$

Cross Correlation

The **cross-correlation function** of $X(t)$ and $Y(t)$ is

$$R_{X,Y}(t_1, t_2) = \mathbb{E}\left[X(t_1)Y(t_2)\right].$$

The **cross-covariance function** of $X(t)$ and $Y(t)$ is

$$C_{X,Y}(t_1, t_2) = \mathbb{E}\left[\left(X(t_1) - \mu_X(t_1)\right)\left(Y(t_2) - \mu_Y(t_2)\right)\right]$$

Remark: $C_{X,Y}(t_1, t_2) = R_{X,Y}(t_1, t_2) = \mathbb{E}[X(t_1)Y(t_2)]$ if $\mu_X(t_1) = \mu_Y(t_2) = 0$.

Independent and uncorrelated

Two random processes $X(t)$ and $Y(t)$ are **independent** if for any $t_1, \ldots, t_N$,

$$
\begin{aligned}
&f_{X(t_1),\ldots,X(t_N),Y(t_1),\ldots,Y(t_N)}(x_1, \ldots, x_N, y_1, \ldots, y_N) \\
&\qquad = f_{X(t_1),\ldots,X(t_N)}(x_1, \ldots, x_N) \times f_{Y(t_1),\ldots,Y(t_N)}(y_1, \ldots, y_N).
\end{aligned}
$$

Two random processes are $X(t)$ and $Y(t)$ **uncorrelated** if

$$
\mathbb{E}\left[X(t_1)Y(t_2)\right] = \mathbb{E}\left[X(t_1)\right]\mathbb{E}\left[Y(t_2)\right], \tag{1}
$$

Independent and uncorrelated

$$\text{independent X and Y} \underset{\not\Leftarrow}{\overset{\Rightarrow}{}} \text{uncorrelated X and Y}$$

$$\begin{aligned}
\mathbb{E}\left[X(t_1)Y(t_2)\right] &= \int X(t_1,\xi)Y(t_2,\zeta)f_{X,Y}(\xi,\zeta)d\xi d\zeta \\
&= \int X(t_1,\xi)Y(t_2,\zeta)f_X(\xi)f_Y(\zeta)d\xi d\zeta, \qquad \text{independence} \\
&= \int X(t_1,\xi)f_X(\xi)d\xi \cdot \int Y(t_2,\zeta)f_Y(\zeta)d\zeta \\
&= \mathbb{E}[X(t_1)]\mathbb{E}[Y(t_2)].
\end{aligned}$$

Example

Example 3. Let $Y(t) = X(t) + N(t)$, where $X(t)$ and $N(t)$ are independent. Find $R_{X,Y}(t_1, t_2)$.

Solution.

$$\begin{aligned} R_{X,Y}(t_1, t_2) &= \mathbb{E}\left[X(t_1)Y(t_2)\right] \\ &= \mathbb{E}\left[X(t_1)\left(X(t_2) + N(t_2)\right)\right] \\ &= R_X(t_1, t_2) + \mu_X(t_1)\mu_N(t_2). \end{aligned}$$

Questions?