ECE 302: Lecture A.5 Wide Sense Stationary Processes

Prof Stanley Chan

School of Electrical and Computer Engineering
Purdue University
Wide Sense Stationary Processes

Definition
A random process $X(t)$ is **wide sense stationary (W.S.S.)** if:

1. $\mu_X(t) = \text{constant}$, for all $t$,
2. $R_X(t_1, t_2) = R_X(t_1 - t_2)$ for all $t_1, t_2$.

**Remark 1**: WSS processes can also be defined using the autocovariance function

$$C_X(t_1, t_2) = C_X(t_1 - t_2).$$

**Remark 2**: Because a WSS is completely characterized by the difference $t_1 - t_2$, there is no need to keep track of the absolute indices $t_1$ and $t_2$. We can rewrite the autocorrelation function as

$$R_X(\tau) = \mathbb{E}[X(t + \tau)X(t)].$$  \hspace{1cm} (1)
Visualizing WSS Processes

Figure: Cross sections of the autocorrelation function

\[ R_X(t_1, t_2) = \frac{1}{2} \cos \left( \omega (t_1 - t_2) \right). \]
Physical Interpretation of $R_X(\tau)$

Consider the following function:

$$\hat{R}_X(\tau) \overset{\text{def}}{=} \frac{1}{2T} \int_{-T}^{T} X(t + \tau)X(t) \, dt.$$  \hspace{1cm} (2)

This function is the **temporal average** of $X(t + \tau)X(t)$

**How do we understand $\hat{R}_X(\tau)$?**

$\hat{R}_X(\tau)$ is the “un-flipped convolution”, or **correlation**, of $X(\tau)$ and $X(t + \tau)$. 
Correlation vs convolution

Figure: The difference between convolution and correlation. In convolution, the function $X(t)$ is flipped before we compute the result. For correlation, there is no flip.
So what?

Lemma

Let \( \hat{R}_X(\tau) \overset{def}{=} \frac{1}{2T} \int_{-T}^{T} X(t + \tau)X(t)dt \). Then,

\[
E[\hat{R}_X(\tau)] = R_X(\tau).
\] (3)

Proof.

\[
E[\hat{R}_X(\tau)] = \frac{1}{2T} \int_{-T}^{T} E[X(t + \tau)X(t)] dt \\
= \frac{1}{2T} \int_{-T}^{T} R_X(\tau) dt \\
= R_X(\tau) \frac{1}{2T} \int_{-T}^{T} dt \\
= R_X(\tau).
\]
Example

**Example 1.** Consider a random process $X(t)$ such that for every $t$, $X(t)$ is an i.i.d. Gaussian random variance with zero mean and unit variance. Find $R_X(\tau)$.

**Solution.**

$$R_X(\tau) = \mathbb{E}[X(t + \tau)X(t)] = \begin{cases} \mathbb{E}[X^2(t)], & \tau = 0, \\ \mathbb{E}[X(t + \tau)]\mathbb{E}[X(t)], & \tau \neq 0 \end{cases}$$

Using the fact that $X(t)$ is i.i.d. Gaussian for all $t$, we can show $\mathbb{E}[X^2(t)] = 1$ for any $t$, and $\mathbb{E}[X(t + \tau)]\mathbb{E}[X(t)] = 0$. Therefore, we have

$$R_X(\tau) = \begin{cases} 1, & \tau = 0, \\ 0, & \tau \neq 0 \end{cases}.$$
Figure: The difference between convolution and correlation. In convolution, the function $X(t)$ is flipped before we compute the result. For correlation, there is no flip.
Properties of $R_X(\tau)$

**Corollary**

$R_X(0) = \text{average power of } X(t)$

**Proof.** Since $R_X(0) = \mathbb{E}[X(t + 0)X(t)] = \mathbb{E}[X(t)^2]$, and since $\mathbb{E}[X(t)^2]$ is the average power, we have that $R_X(0)$ is the average power of $X(t)$.

**Corollary**

$R_X(\tau)$ is symmetric. That is, $R_X(\tau) = R_X(-\tau)$.

**Proof.** Note that $R_X(\tau) = \mathbb{E}[X(t + \tau)X(t)]$. By switching the order of multiplication in the expectation, we have $\mathbb{E}[X(t + \tau)X(t)] = \mathbb{E}[X(t)X(t + \tau)] = R_X(-\tau)$. 
Corollary

\[ \mathbb{P}(|X(t + \tau) - X(\tau)| > \epsilon) \leq \frac{2(R_X(0) - R_X(\tau))}{\epsilon^2} \]

This result says that if \( R_X(\tau) \) is slowly decaying from \( R_X(0) \), then the probability of having a large deviation \(|X(t + \tau) - X(\tau)|\) is small.

**Proof.**

\[
\begin{align*}
\mathbb{P}(|X(t + \tau) - X(\tau)| > \epsilon) &\leq \frac{\mathbb{E}[(X(t + \tau) - X(\tau))^2]}{\epsilon^2} \\
&= \left( \mathbb{E}[X(t + \tau)^2] - 2\mathbb{E}[X(t + \tau)X(t)] + \mathbb{E}[X(t)^2] \right) / \epsilon^2 \\
&= \left( 2\mathbb{E}[X(t)^2] - 2\mathbb{E}[X(t + \tau)X(t)] \right) / \epsilon^2 \\
&= 2 \left( R_X(0) - R_X(\tau) \right) / \epsilon^2.
\end{align*}
\]
Properties of $R_X(\tau)$

**Corollary**

$|R_X(\tau)| \leq R_X(0)$, for all $\tau$.

**Proof.** By Cauchy inequality $\mathbb{E}[XY]^2 \leq \mathbb{E}[X^2]\mathbb{E}[Y^2]$, we can show that

\[
R_X(\tau)^2 = \mathbb{E}[X(t)X(t+\tau)]^2 \\
\leq \mathbb{E}[X(t)^2] \mathbb{E}[X(t+\tau)^2] \\
= \mathbb{E}[X(t)^2]^2 \\
= R_X(0)^2.
\]
Ergodic theorem

Under what conditions will $\hat{R}_X(\tau) \to R_X(\tau)$ as $T \to \infty$?

**Theorem (Mean-Square Ergodic Theorem)**

Let $Y(t)$ be a W.S.S. process, with mean $\mathbb{E}[Y(t)] = \mu$ and autocorrelation function $R_Y(\tau)$. Define

$$M_T \overset{\text{def}}{=} \frac{1}{2T} \int_{-T}^{T} Y(t)dt. \quad (4)$$

Then, $\mathbb{E} \left[ |M_T - \mu|^2 \right] \to 0$ as $T \to \infty$ if and only if

$$\lim_{T \to \infty} \left[ \frac{1}{2T} \int_{-T}^{T} \left(1 - \frac{\tau}{2T}\right) R_Y(\tau) d\tau \right] = 0. \quad (5)$$

**Proof** Optional. See eBook.
Everything you need to know about a WSS process.

- The mean of a WSS process is a constant (does not need to be zero)
- The correlation function only depends on the difference, so $R_X(t_1, t_2)$ is toeplitz.
- You can write $R_X(t_1, t_2)$ as $R_X(\tau)$, where $\tau = t_1 - t_2$.
- $R_X(\tau)$ tells you how much correlation you have with someone located at a time instant $\tau$ from you.
- You can think of $R_X(\tau)$ as the temporal correlation $\hat{R}_X(\tau)$.
- Under certain regularity conditions, $\hat{R}_X(\tau)$ is a good approximation of $R_X(\tau)$. 
Questions?