ECE 302: Lecture A.5 Wide Sense Stationary Processes

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Wide Sense Stationary Processes

Definition

A random process $X(t)$ is **wide sense stationary (W.S.S.)** if:

1. $\mu_X(t) = \text{constant}, \text{ for all } t,$
2. $R_X(t_1, t_2) = R_X(t_1 - t_2)$ for all t_1, t_2.

Remark 1: WSS processes can also be defined using the autocovariance function

$$C_X(t_1, t_2) = C_X(t_1 - t_2).$$

Remark 2: Because a WSS is completely characterized by the difference $t_1 - t_2$, there is no need to keep track of the absolute indices t_1 and t_2. We can rewrite the autocorrelation function as

$$R_X(\tau) = \mathbb{E}[X(t+\tau)X(t)]. \tag{1}$$

Visualizing WSS Processes

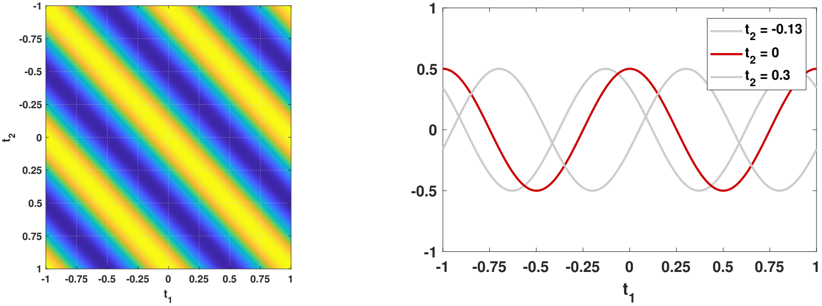


Figure: Cross sections of the autocorrelation function
$R_X(t_1, t_2) = \frac{1}{2}\cos\Big(\omega(t_1 - t_2)\Big)$.

Physical Interpretation of $R_X(\tau)$

Consider the following function:

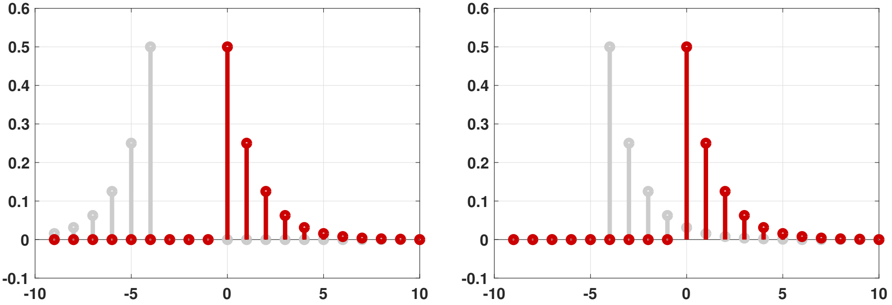
$$\widehat{R}_X(\tau) \overset{\text{def}}{=} \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)dt. \tag{2}$$

This function is the **temporal average** of $X(t+\tau)X(t)$

How do we understand $\widehat{R}_X(\tau)$?

$\widehat{R}_X(\tau)$ is the "un-flipped convolution", or **correlation**, of $X(\tau)$ and $X(t+\tau)$.

Correlation vs convolution



(a) Convolution (b) Correlation

Figure: The difference between convolution and correlation. In convolution, the function $X(t)$ is flipped before we compute the result. For correlation, there is no flip.

So what?

Lemma

Let $\widehat{R}_X(\tau) \overset{def}{=} \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)dt$*. Then,*

$$\mathbb{E}\left[\widehat{R}_X(\tau)\right] = R_X(\tau). \tag{3}$$

Proof.

$$\begin{aligned}
\mathbb{E}\left[\widehat{R}_X(\tau)\right] &= \frac{1}{2T}\int_{-T}^{T} \mathbb{E}\left[X(t+\tau)X(t)\right] dt \\
&= \frac{1}{2T}\int_{-T}^{T} R_X(\tau)dt \\
&= R_X(\tau)\frac{1}{2T}\int_{-T}^{T} dt \\
&= R_X(\tau).
\end{aligned}$$

Example

Example 1. Consider a random process $X(t)$ such that for every t, $X(t)$ is an i.i.d. Gaussian random variance with zero mean and unit variance. Find $R_X(\tau)$.

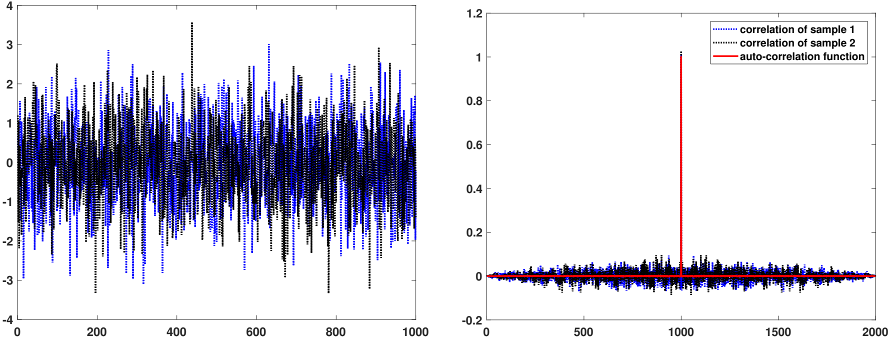
Solution.

$$R_X(\tau) = \mathbb{E}[X(t+\tau)X(t)] = \begin{cases} \mathbb{E}[X^2(t)], & \tau = 0, \\ \mathbb{E}[X(t+\tau)]\mathbb{E}[X(t)], & \tau \neq 0 \end{cases}$$

Using the fact that $X(t)$ is i.i.d. Gaussian for all t, we can show $\mathbb{E}[X^2(t)] = 1$ for any t, and $\mathbb{E}[X(t+\tau)]\mathbb{E}[X(t)] = 0$. Therefore, we have

$$R_X(\tau) = \begin{cases} 1, & \tau = 0, \\ 0, & \tau \neq 0 \end{cases}.$$

Visualization



(a) $X(t)$ (b) $\widehat{R}_X(\tau)$

Figure: The difference between convolution and correlation. In convolution, the function $X(t)$ is flipped before we compute the result. For correlation, there is no flip.

Properties of $R_X(\tau)$

Corollary

$R_X(0) =$ *average power of* $X(t)$

Proof. Since $R_X(0) = \mathbb{E}[X(t+0)X(t)] = \mathbb{E}[X(t)^2]$, and since $\mathbb{E}[X(t)^2]$ is the average power, we have that $R_X(0)$ is the average power of $X(t)$.

Corollary

$R_X(\tau)$ *is symmetric. That is,* $R_X(\tau) = R_X(-\tau)$.

Proof. Note that $R_X(\tau) = \mathbb{E}[X(t+\tau)X(t)]$. By switching the order of multiplication in the expectation, we have
$\mathbb{E}[X(t+\tau)X(t)] = \mathbb{E}[X(t)X(t+\tau)] = R_X(-\tau)$.

Properties of $R_X(\tau)$

Corollary

$$\mathbb{P}(|X(t+\tau) - X(\tau)| > \epsilon) \leq \frac{2(R_X(0) - R_X(\tau))}{\epsilon^2}$$

This result says that if $R_X(\tau)$ is slowly decaying from $R_X(0)$, then the probability of having a large deviation $|X(t+\tau) - X(\tau)|$ is small.

Proof.

$$\begin{aligned}
\mathbb{P}(|X(t+\tau) - X(\tau)| > \epsilon) &\leq \mathbb{E}[(X(t+\tau) - X(\tau))^2]/\epsilon^2 \\
&= \Big(\mathbb{E}[X(t+\tau)^2] - 2\mathbb{E}[X(t+\tau)X(t)] + \mathbb{E}[X(t)^2]\Big) \\
&= \Big(2\mathbb{E}[X(t)^2] - 2\mathbb{E}[X(t+\tau)X(t)]\Big)/\epsilon^2 \\
&= 2\Big(R_X(0) - R_X(\tau)\Big)/\epsilon^2.
\end{aligned}$$

Properties of $R_X(\tau)$

Corollary

$|R_X(\tau)| \leq R_X(0)$, *for all* τ.

Proof. By Cauchy inequality $\mathbb{E}[XY]^2 \leq \mathbb{E}[X^2]\mathbb{E}[Y^2]$, we can show that

$$\begin{aligned} R_X(\tau)^2 &= \mathbb{E}[X(t)X(t+\tau)]^2 \\ &\leq \mathbb{E}[X(t)^2]\mathbb{E}[X(t+\tau)^2] \\ &= \mathbb{E}[X(t)^2]^2 \\ &= R_X(0)^2. \end{aligned}$$

Ergodic theorem

Under what conditions will $\widehat{R}_X(\tau) \to R_X(\tau)$ as $T \to \infty$?

Theorem (Mean-Square Ergodic Theorem)

Let $Y(t)$ be a W.S.S. process, with mean $\mathbb{E}[Y(t)] = \mu$ and autocorrelation function $R_Y(\tau)$. Define

$$M_T \stackrel{def}{=} \frac{1}{2T}\int_{-T}^{T} Y(t)dt. \tag{4}$$

Then, $\mathbb{E}\left[\left|M_T - \mu\right|^2\right] \to 0$ as $T \to \infty$ if and only if

$$\lim_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T}\left(1 - \frac{|\tau|}{2T}\right)R_Y(\tau)d\tau\right] = 0. \tag{5}$$

Proof Optional. See eBook.

Summary

Everything you need to know about a WSS process.

- The mean of a WSS process is a constant (does not need to be zero)
- The correlation function only depends on the difference, so $R_X(t_1, t_2)$ is toeplitz.
- You can write $R_X(t_1, t_2)$ as $R_X(\tau)$, where $\tau = t_1 - t_2$.
- $R_X(\tau)$ tells you how much correlation you have with someone located at a time instant τ from you.
- You can think of $R_X(\tau)$ as the temporal correlation $\widehat{R}_X(\tau)$.
- Under certain regularity conditions, $\widehat{R}_X(\tau)$ is a good approximation of $R_X(\tau)$.

Questions?