ECE 302: Lecture A.7 Review of LTI Systems

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Random process through a system

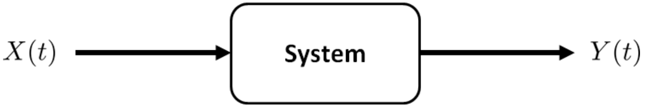

Figure: A system can be viewed as a blackbox that takes an input $X(t)$ and turns it into an output $Y(t)$.

- Random processes have limited usefulness until we can apply operations to them.
- We discuss how WSS processes respond to a **linear time invariant** (LTI) system.
- Useful in signal processing, communication, speech analysis, and imaging.

Example of Running Average

Linearity

- **Linearity**. Linearity says that when two input random processes can be added and scaled, then the output random processes will be added and shifted in the exact same way. Mathematically, linearity says that if $X_1(t) \rightarrow Y_1(t)$ and $X_2(t) \rightarrow Y_2(t)$, then

$$aX_1(t) + bX_2(t) \rightarrow aY_1(t) + bY_2(t).$$

Time Invariant

- **Time invariant**: Time invariance says that if we shift the input random process by certain time period, the output will be shifted in the same way. Mathematically, time invariance means that if $X(t) \to Y(t)$, then

$$X(t+\tau) \to Y(t+\tau).$$

Convolution

The **convolution** between two functions $X(t)$ and $h(t)$ is defined as

$$Y(t) = h(t) * X(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau,$$

where we call $h(t)$ as the system response or **impulse response**.

$h(t)$ is called the impulse response because if $X(t) = \delta(t)$, then according to the convolution equation we will have

$$Y(t) = \int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)d\tau = h(t).$$

Therefore, if we send an impulse to the system, the output will be $h(t)$.

Convolution is commutative

Convolution is commutative, meaning that

$$h(t) * X(t) = X(t) * h(t)$$

Written in the integrations, we have

$$\int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau = \int_{-\infty}^{\infty} h(t-\tau)X(\tau)d\tau. \quad (1)$$

Fourier Transform

For linear time-invariant systems, the output $Y(t)$ can be determined through the Fourier transforms.

The **Fourier transform** of a (squared-integrable) function $X(t)$ is

$$X(\omega) = \mathcal{F}\{X(t)\} = \int_{-\infty}^{\infty} X(\tau)e^{-j\omega\tau}d\tau \tag{2}$$

Three important things about Fourier transforms

- Fourier transform is an orthogonal mapping
- $e^{j\omega t}$ is an eigen function
- Convolution in t = multiplication in ω

Fourier transform is an orthogonal mapping

$e^{j\omega t}$ is an eigen-function

Let $x(t) = e^{j\omega t}$. Then, when sent through an LTI system:

$$\begin{aligned} y(t) = h(t) * x(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau e^{j\omega t} \\ &= H(\omega)e^{j\omega t} = H(\omega)x(t). \end{aligned}$$

Convolution in t = Multiplication in ω

A basic property of the convolution is that convolution in the time domain is equivalent to multiplication in the Fourier domain. Therefore,

$$Y(\omega) = H(\omega)X(\omega), \tag{3}$$

where $H(\omega) = \mathcal{F}\{h(t)\}$ is the Fourier transform of $h(t)$, and $Y(\omega) = \mathcal{F}(Y(t))$ is the Fourier transform of $Y(t)$.

Summary

- Much of this lecture can be found in a standard textbook on Signals and Systems.
- Alan Oppenheim and Alan Willsky, *Signals and Systems*, Pearson, 2nd edition, 1996.
- The concept is applicable to applications beyond EE/CS.

In the rest of this chapter we study the pair of random processes.

- $X(t)$ = input. It is a WSS random process.
- $Y(t)$ = output. It is constructed by sending $X(t)$ through an LTI system with impulse response $h(t)$. Therefore, $Y(t) = h(t) * X(t)$.

Questions?