

ECE 302: Lecture 1.2 Approximation

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Outline

- 1.1 Infinite Series
 - 1.1.1. Geometric Series
 - 1.1.2. Binomial Series
- 1.2 Approximations
 - 1.2.1. Taylor Approximation
 - 1.2.2. Exponential Series
 - 1.2.3. Logarithmic Approximation
- 1.3 Integration
 - 1.3.1. Odd and Even Functions
 - 1.3.2. Fundamental Theorem of Calculus
- 1.4 Linear Algebra (Optional)
 - 1.4.1. Inner Products (Optional)
 - 1.4.2. Matrix Calculus (Optional)
 - 1.4.3. Matrix Inversion (Optional)
- 1.5 Combinatorics
 - 1.5.1. Permutation
 - 1.5.2. Combination

Taylor Approximation

Definition

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with infinite derivatives. Let $a \in \mathbb{R}$ be a fixed constant. The Taylor approximation of f at $x = a$ is

$$\begin{aligned} f(x) &= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n, \end{aligned}$$

where $f^{(n)}$ denotes the n -th order derivative of f .

Example 1

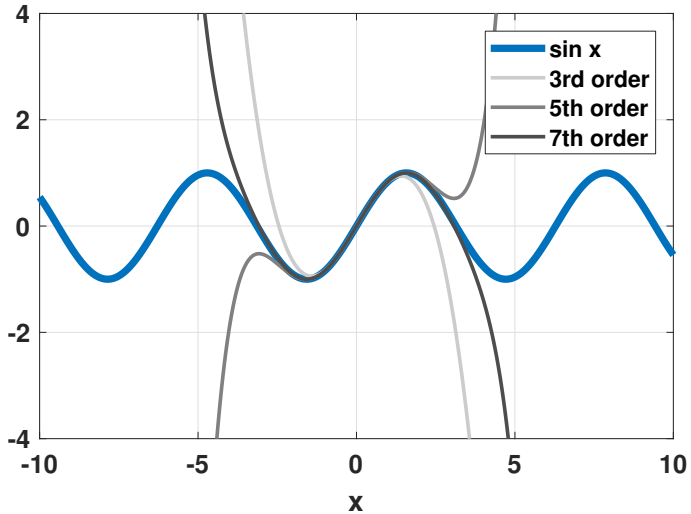
Find Taylor approximation of $f(x) = \sin x$ at $x = 0$.

Solution. The Taylor approximation at $x = 0$ is

$$\begin{aligned}f(x) &\approx f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \frac{f'''(0)}{3!}(x - 0)^3 \\&= \sin(0) + (\cos 0)(x - 0) - \frac{\sin(0)}{2!}(x - 0)^2 - \frac{\cos(0)}{3!}(x - 0)^3 \\&= x - \frac{x^3}{6}.\end{aligned}$$

We can expand further to higher orders, which yields

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

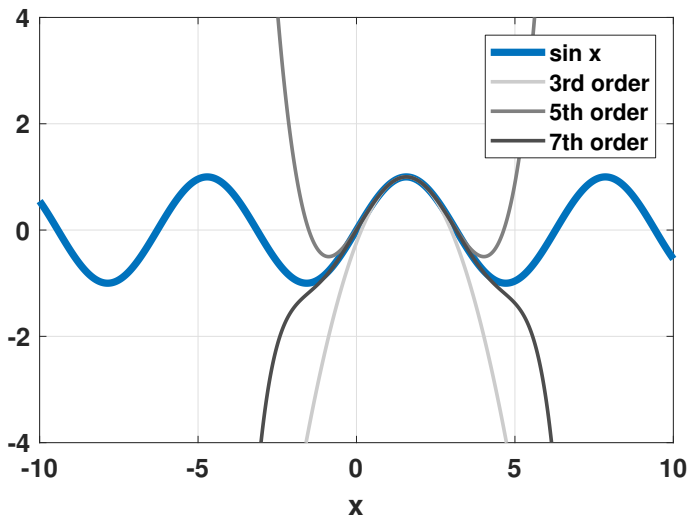


Example 2

Find Taylor approximation of $f(x) = \sin x$ at $x = \pi/2$.

Solution. Taylor approximation at $x = \pi/2$ for $f(x) = \sin x$ is

$$\begin{aligned}f(x) &= \sin \frac{\pi}{2} + \cos \frac{\pi}{2} \left(x - \frac{\pi}{2}\right) - \frac{\sin \frac{\pi}{2}}{2!} \left(x - \frac{\pi}{2}\right)^2 - \frac{\cos \frac{\pi}{2}}{3!} \left(x - \frac{\pi}{2}\right)^3 \\ &= 1 - \frac{1}{4} \left(x - \frac{\pi}{2}\right)^2.\end{aligned}$$



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Exponential Series

Theorem

Let x be any real number. Then,

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \quad (1)$$

Proof. Let $f(x) = e^x$ for any x . Then, Taylor approximation at $x = 0$ is

$$\begin{aligned} f(x) &= f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \dots \\ &= e^0 + e^0(x - 0) + \frac{e^0}{2!}(x - 0)^2 + \dots \\ &= 1 + x + \frac{x^2}{2} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \end{aligned}$$

This result will be used in **Poisson** random variables.

Example 1

Evaluate the sum $\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}$.

Example 2

Substitute $x = j\theta$ where $j = \sqrt{-1}$. Find the Taylor approximation of sine and cosine.

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Logarithmic Approximation

Theorem

Let $0 < x < 1$ be a constant. Then,

$$\log(1+x) = x - x^2 + \mathcal{O}(x^3). \quad (2)$$

Proof. Let $f(x) = \log(1+x)$. Then, the derivatives of f are

$$f'(x) = \frac{1}{(1+x)}, \quad \text{and} \quad f''(x) = -\frac{1}{(1+x)^2}.$$

Taylor approximation at $x = 0$ gives

$$\begin{aligned} f(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \mathcal{O}(x^3) \\ &= x - x^2 + \mathcal{O}(x^3). \end{aligned}$$

This result will be used in **Central Limit Theorem**.

Example

Show that

$$\lim_{N \rightarrow \infty} \left(1 + \frac{s^2}{2N}\right)^N = e^{s^2/2}. \quad (3)$$

Questions?