ECE 302: Lecture 2.6 Bayes Theorem and Total Probability

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Outline

Prisoner's Dilemma

- Three Prisoners: A, B, C. The King decides to release 2 and sentence 1.
- You were A.
- Your chance of release is $2/3$.
- Suppose you know the guard well. You can ask him about which of B or C will be released.
- But if you find out B (or C) is released, your chance becomes $1/2$.
- How come!!

Conditional Probability

- X_A = sentence A, X_B = sentence B, X_C = sentence C.
- G_B = guard says B is released.

Let us find the probabilities:

- $\mathbb{P}[X_A]$, $\mathbb{P}[X_B]$, $\mathbb{P}[X_C]$

Conditional Probability

- $X_A =$ sentence A, $X_B =$ sentence B, $X_C =$ sentence C.
- $G_B =$ guard says B is released.

Let us find the probabilities:

- $\mathbb{P}[G_B \mid X_A]$, $\mathbb{P}[G_B \mid X_B]$, $\mathbb{P}[G_B \mid X_C]$

Stuck...

$$\mathbb{P}[X_A \mid G_B] =$$

Outline

Bayes Theorem

Theorem (Bayes Theorem)

For any two events A and B such that $\mathbb{P}[A] > 0$ *and* $\mathbb{P}[B] > 0$*, it holds that*

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[B \mid A]\,\mathbb{P}[A]}{\mathbb{P}[B]}.$$

Law of Total Probability

Theorem (**Law of Total Probability**)

Let $\{A_1, A_2, \ldots, A_n\}$ *be a partition of* Ω*, i.e.,* $A_1, \ldots, A_n$ *are disjoint and* $\Omega = A_1 \cup A_2 \cup \ldots \cup A_n$*. Then, for any* $B \subseteq \Omega$*,*

$$\mathbb{P}[B] = \sum_{i=1}^{n} \mathbb{P}[B \mid A_i]\, \mathbb{P}[A_i].$$

Law of Total Probability

continue...

Law of Total Probability

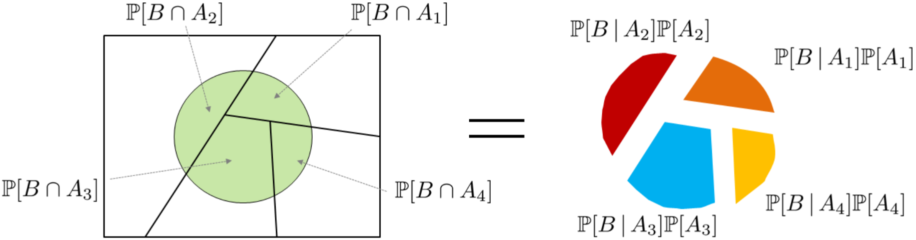


Figure: Law of total probability decomposes the probability $\mathbb{P}[B]$ into multiple conditional probabilities $\mathbb{P}[B \mid A_i]$. The probability of obtaining each $\mathbb{P}[B \mid A_i]$ is $\mathbb{P}[A_i]$.

Outline

Example 1: Tennis Tournament

Example. Consider a tennis tournament. Your probability of winning the game is

$$0.3 \text{ against } \frac{1}{2} \text{ of the players (Event A).}$$
$$0.4 \text{ against } \frac{1}{4} \text{ of the players (Event B).}$$
$$0.5 \text{ against } \frac{1}{4} \text{ of the players (Event C).}$$

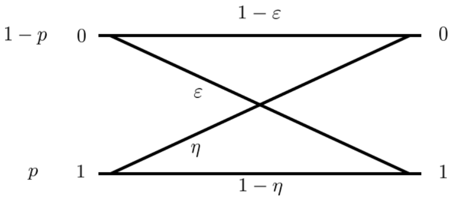
What is the probability of winning the game?

Example 1

continue...

Example 2: Communication Channel

Example. Consider a communication channel shown below. The probability of sending a 1 is p and the probability of sending a 0 is $1 - p$. Given that 1 is sent, the probability of receiving 1 is $1 - \eta$. Given that 0 is sent, the probability of receiving 0 is $1 - \varepsilon$.



- Find $\mathbb{P}$[Receive 1]
- Find $\mathbb{P}$[Send 1]
- Find $\mathbb{P}$[Receive 1 | Send 1]
- Find $\mathbb{P}$[Send 1 | Receive 1]

Example 2

continue...

Example 2

continue...

Prisoner's Dilemma (continue)

$$\mathbb{P}[X_A \mid G_B] =$$

Questions?