ECE 302: Lecture 3.1 Random Variables

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Outline

- 3.1 Random variables
  - What are random variables?
  - What is the difference between a variable and a random variable?
  - Why do we need random variables?
- 3.2 Probability mass functions
- 3.3 Cumulative distribution functions
- 3.4 Expectation
- 3.5 Moments and variance
- 3.6 Bernoulli random variables
- 3.7 Binomial random variables
- 3.8 Geometric random variables
- 3.9 Poisson random variables

Mathematicians are lazy

Example 1: Let us flip a coin.

$$\text{“probability of getting a head”} = \mathbb{P}[\text{“Head”}]$$

... Easy!

Example 2: Let us flip 3 coins.

$$\text{“probability of getting 3 heads”} = \mathbb{P}[\text{“Head”} \cap \text{“Head”} \cap \text{“Head”}].$$

... Err...

Mathematicians say: Call “Head” $= 1$ and “Tail” $= 0$. Let X be the sum of heads. Then

$$\text{“probability of getting a head”} = \mathbb{P}[X = 1]$$

$$\text{“probability of getting 3 heads”} = \mathbb{P}[X = 3]$$

... Life is good!

Everything you need to know about a random variable

Question: What are random variables?
Answer: Random variables are **functions** that translate words to numbers.
Example: "Head" is a word description. "X = 1" is numerical description.

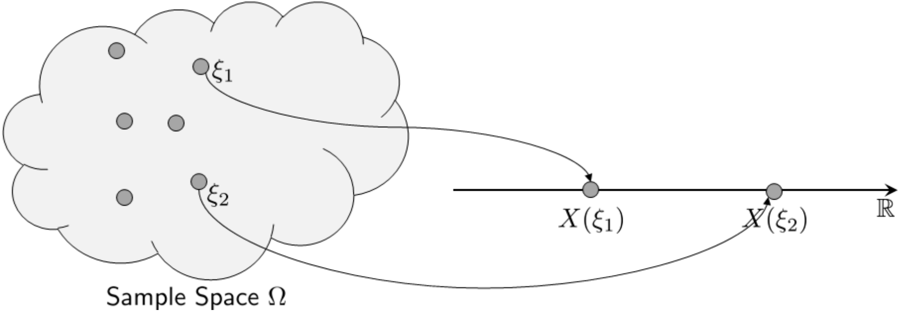


Figure: Illustration of a random variable.

The formal definition

Definition

A **random variable** X is a function $X : \Omega \to \mathbb{R}$ that maps an outcome $\xi \in \Omega$ to a number $X(\xi)$ on the real line.

Why need Random Variable?

- Coin flip: Head or Tail
- Alphabet: a, b, c, ..., z

We want to map these outcomes to numbers.

Random variables are functions that translate words to numbers!

Example

- Flip a coin 2 times.
- The sample space Ω is

$$\Omega = \{(\mathrm{HH}), (\mathrm{HT}), (\mathrm{TH}), (\mathrm{TT})\}.$$

- Four events $\xi_1 = \mathrm{HH}$, $\xi_2 = \mathrm{HT}$, $\xi_3 = \mathrm{TH}$, $\xi_4 = \mathrm{TT}$.

Then,

- $X =$ number of H.
- $X(\xi_1) = \#$ of heads in $\{HH\} = 2$
- $X(\xi_2) = \#$ of heads in $\{HT\} = 1$
- $X(\xi_3) = \#$ of heads in $\{TH\} = 1$
- $X(\xi_4) = \#$ of heads in $\{TT\} = 0$

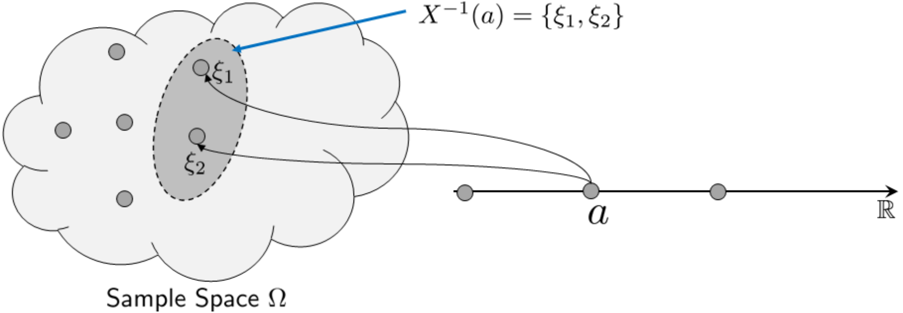
Calculating the probability

How to "calculate" $\mathbb{P}[X=1]$?

- Is $\{X=1\}$ an event in the event space?
- No. The sample space is
$$\Omega = \{(\text{HH}), (\text{HT}), (\text{TH}), (\text{TT})\}.$$
- The event space is the combination of these outcomes.
- "$\{X=1\}$" is not the same as HT or TH.
- "$\{X=1\}$" lives in the **translated** space.
- So how to measure the probability?
- Map $\{X=1\}$ back to the sample space!
$$\mathbb{P}[\{X=1\}] = \mathbb{P}[\{(\text{HT}) \cup (\text{TH})\}] = \frac{2}{4}. \tag{1}$$
- Essentially, you are finding ξ such that $X(\xi)=1$. So,
$$\mathbb{P}[X=1] = \mathbb{P}[X(\xi)=1] = \mathbb{P}[\xi = X^{-1}(1)] = \mathbb{P}[\{(\text{HT}) \cup (\text{TH})\}] = \frac{2}{4}.$$

The inverse mapping

When calculating the probability, go backward to the sample space!



- Important! $X^{-1}(a)$ is a set; it is an event in the event space.
- You can measure an event using $\mathbb{P}$, but you cannot measure the event in the translated space.

Example 1

- Flip a coin 2 times.
- The sample space Ω is

$$\Omega = \{(\mathrm{HH}), (\mathrm{HT}), (\mathrm{TH}), (\mathrm{TT})\}.$$

- $\xi_1 = \mathrm{HH}$, $\xi_2 = \mathrm{HT}$, $\xi_3 = \mathrm{TH}$, $\xi_4 = \mathrm{TT}$.
- $X =$ number of H.

What is the probability of getting 1 head?

- What is the sample space?
- What is X? What is a?
- What is $X^{-1}(a)$?
- What are the elements we need to count?

Example 2

- Throw a dice 2 times.
- The sample space Ω is

$$\Omega = \{(1,1),(1,2),\ldots,(6,6)\}.$$

- $\xi_1 = (1,1)$, $\xi_2 = (1,2)$, ..., $\xi_{36} = (6,6)$.
- $X =$ sum of two numbers.

What is the probability of getting a 7?

- What is the sample space?
- What is X? What is a?
- What is $X^{-1}(a)$?
- What are the elements we need to count?

Summary

- A random variable is a **function** that translate words to numbers
- To measure $X = a$, we take the **inverse image** to find the actual event in the event space $X^{-1}(a)$.
- Then apply probability measure to the set $X^{-1}(a)$
- In practice, you do not need to worry about these.
- $X =$ dice, then $\mathbb{P}[X = 1] = \frac{1}{6}$. You never do this translation per se.

Questions?