# ECE 302: Lecture 3.2 Probability Mass Functions

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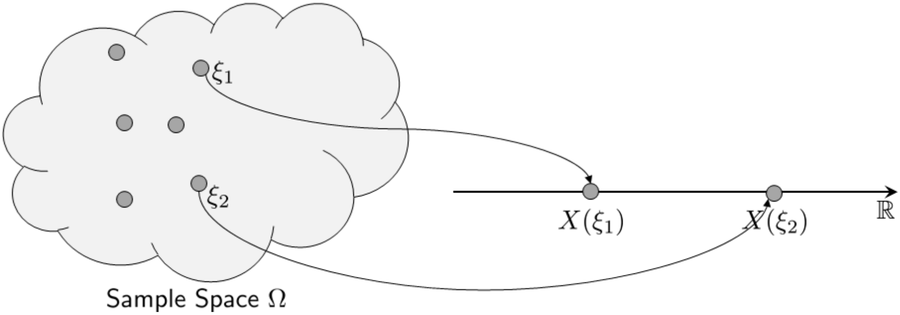
# Outline

# Everything you need to know about random variables

**Random variables are functions that translate words to numbers!**




- So many numbers
- So many events
- How to systematically describe them?

# Probability Mass Function

**Definition**

The **probability mass function (PMF)** of a random variable $X$ is a function which specifies the probability of obtaining a number $X(\xi) = a$. We denote a PMF as

$$p_X(a) = \mathbb{P}[X = a].$$

- There are two functions here:
  - Function $X$: the random variable which translates words to numbers
  - Function $p_X$: the mapping from event $\{X = a\}$ to a probability
- Difference between $X$ and $a$:
  - $X$ is the **random variable**. Technically it should be $X(\xi)$
  - $a$ is a **state**. So $X = a$ means $X$ is taking the state $a$.

**A random variable is random because it has many states!**

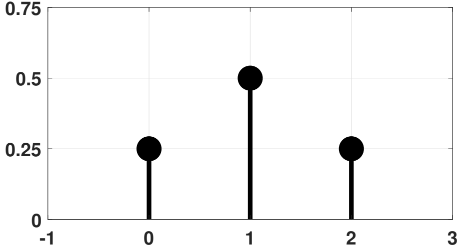
# Example

Flip a coin twice. Define $X$ = number of heads. Then the probability mass function is

$$p_X(0) = \mathbb{P}[X = 0] = \mathbb{P}[\{\text{"TT"}\}] = \frac{1}{4},$$

$$p_X(1) = \mathbb{P}[X = 1] = \mathbb{P}[\{\text{"TH"}, \text{"HT"}\}] = \frac{1}{2},$$

$$p_X(2) = \mathbb{P}[X = 2] = \mathbb{P}[\{\text{"HH"}\}] = \frac{1}{4}.$$



# Difference between variable and random variable

Solve an equation $2X + a = 0$.

- If $a$ is fixed, then $X$ is an ordinary variable
- If $a$ has multiple states, and is random, then $X$ is a random variable.

| | Ordinary Variable | Random Variable |
|---|---|---|
| Equation | $2X + 1 = 0$ | $2X + 1 = 0$ or<br>$2X + 0 = 0$ or<br>$2X - 1 = 0$ |
| Solution | $X = -\frac{1}{2}$ | $X = +\frac{1}{2}$ or<br>$X = 0$ or<br>$X = -\frac{1}{2}$ |
| Random? | No | Yes |
| No. of states | 1 (deterministic) | 3 |
| PMF | Not applicable | $p_X(x) = \frac{1}{3}$, for $x = \frac{1}{2}, 0, -\frac{1}{2}$ |

# Outline

# Histogram

We all know what histograms are.

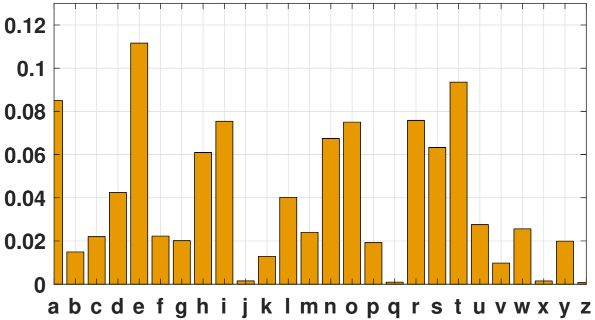
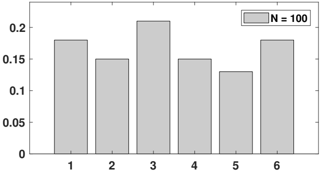



Figure: The frequency of the 26 English letters. Data source: Wikipedia.

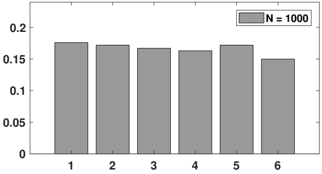
- A histogram contains **state**, and their **probability**.

# Histogram VS PMF

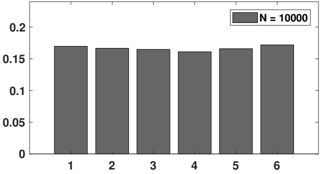
**Question**: What is the difference between a PMF and a histogram?
**Answer**: PMF is the **ideal** histogram!



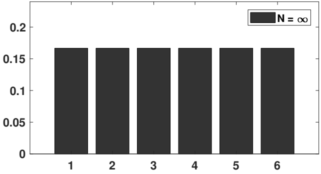

(a) $N = 100$




(b) $N = 1000$




(c) $N = 10000$

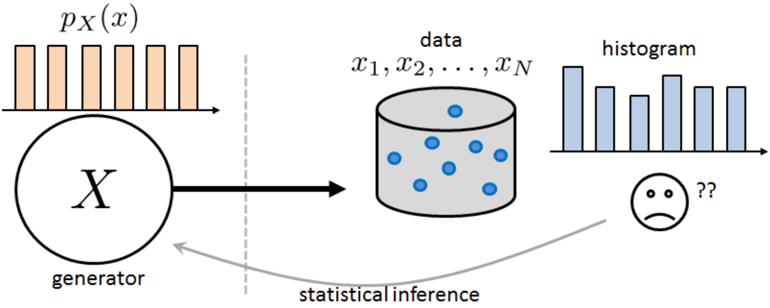



(d) PMF

Figure: Histogram and PMF, when throwing a fair dice $N$ times. As $N$ increases, the histograms are becoming more like the PMF.

# Why border to study PMF?

- Ideal vs empirical. You always want something ideal.
- Modeling the data. Histogram is non-parametric.
- Generator. Synthesize data, and test whether your algorithm will give what you want.



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# Properties of PMF

**Theorem**

*A PMF should satisfy the condition that*

$$\sum_{x \in X(\Omega)} p_X(x) = 1. \tag{1}$$

Proof.

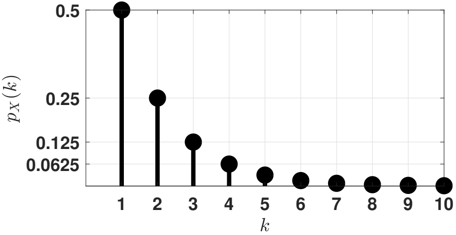
$$\begin{aligned}\sum_{x \in X(\Omega)} \mathbb{P}[X = x] &= \sum_{x \in X(\Omega)} \mathbb{P}\left[\{\xi \in \Omega \,|\, X(\xi) = x\}\right] \\ &= \mathbb{P}\left[\bigcup_{\xi \in \Omega} \{\xi \in \Omega \,|\, X(\xi) = x\}\right] = \mathbb{P}[\Omega] = 1. \qquad \square\end{aligned}$$

# Examples

**Example 1**. Let $X$ be a random variable with PMF

$$p_X(k) = c\left(\frac{1}{2}\right)^k, \qquad k = 1, 2, \ldots$$

Find $c$.

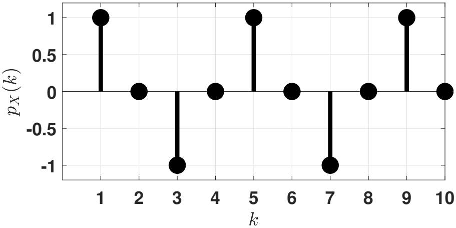


# Examples

**Example 2**. Let $X$ be a random variable with PMF

$$p_X(k) = c \sin\left(\frac{\pi}{2}k\right),$$

for $k = 0, 1, 2, \ldots$. Find $c$.



**Questions?**