ECE 302: Lecture 3.4 Expectation

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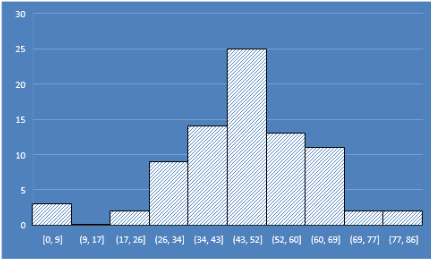
Long long time ago, in a galaxy far far away

Some kids on another planet took a mid term exam!

Here were their scores...

Student 1	86.00
Student 2	76.20
Student 3	29.10
Student 4	26.38
Student 5	60.86
...	...
Student 71	48.04
Student 72	30.20
Student 73	55.44
Student 74	49.92
Student 75	17.60



The teacher felt that the scores are too low, and decided to give 10 points to everyone.

- Will the mean change?
- Will the standard deviation change?
- If the letter grades are curved, will this change the grades?

Outline

- 3.1 Random variables
- 3.2 Probability mass functions (PMF)
- 3.3 Cumulative distribution functions (discrete case)
- 3.4 Expectation
  - 3.4.1 Understanding expectation
  - 3.4.2 Properties of expectation
- 3.5 Moments and variance
- 3.6 Bernoulli random variables
- 3.7 Binomial random variables
- 3.8 Geometric random variables
- 3.9 Poisson random variables

Definition of expectation

Definition (Expectation)

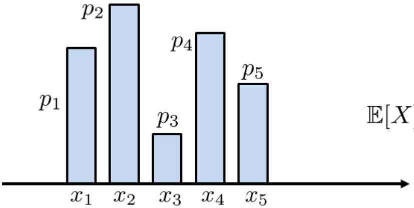
The **expectation** of a random variable X is

$$\mathbb{E}[X] = \sum_{x \in \Omega(X)} x \, p_X(x). \tag{1}$$

Interpretation: Weighted average.

$$\mathbb{E}[X] = \underbrace{\sum_{x \in \Omega(X)}}_{\text{sum over all states}} \underbrace{x}_{\text{a state } X \text{ takes}} \underbrace{p_X(x)}_{\text{the percentage}}.$$

Here is a better way to visualize



$$\mathbb{E}[X] = p_1 x_1 + \ldots + p_5 x_5$$

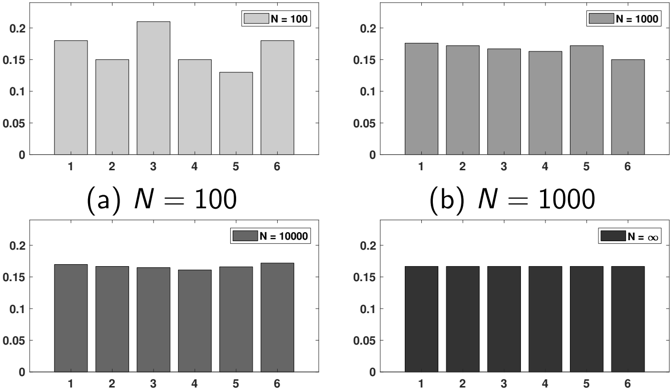
Expectation = "average"

If your dataset is partitioned into several bins, then expectation = "average":

$$\text{average} = \underbrace{\sum_{k=1}^{K}}_{\text{sum of all states}} \underbrace{\text{value } x_k}_{\text{a state } X \text{ takes}} \times \underbrace{\frac{\text{number of samples with value } x_k}{N}}_{\text{the percentage}},$$

Expectation = "average"?

No. Expectation is computed from PMF. Average is computed from histogram.



(a) $N = 100$ (b) $N = 1000$

(c) $N = 10000$ (d) PMF

Figure: Histogram and PMF, when throwing a fair dice N times. As N increases, the histograms are becoming more like the PMF.

Examples

Example 1. Let X be a random variable with PMF $p_X(0) = 1/4$, $p_X(1) = 1/2$ and $p_X(2) = 1/4$. Find $\mathbb{E}[X]$.

Example 2. Flip an unfair coin, where the probability of getting a head is $\frac{3}{4}$. Let X be a random variable such that $X = 1$ means getting a head. Find $\mathbb{E}[X]$.

Example 3. Let X be a random variable with PMF

$$p_X(k) = \frac{c}{2^k}, \qquad k = 1, 2, \ldots$$

(a) Find c
(b) Find $\mathbb{E}[X]$

Examples

Example 4. Consider a game. Flip a coin 3 times. Reward:

- \$1 if there are 2 Heads
- \$8 if there are 3 Heads
- \$0 if there are 0 or 1 Head

The cost to enter the game is \$1.5. On average what is the net gain?

True Mean and Sample Mean

True Mean $\mathbb{E}[X]$

- A statistical property of a random variable.
- A deterministic number.
- Often unknown, or is the center question of estimation.
- You have to know X in order to find $\mathbb{E}[X]$; Top down.

Sample Mean $\overline{X}$

- A numerical value. Calculated from data.
- Itself is a random variable.
- It has uncertainty.
- Uncertainty reduces as more samples are used.
- We use sample mean to estimate the true mean.
- You do not need to know X in order to find $\overline{X}$; Bottom up.

Existence of expectation

Does expectation always exist?

No.

Example. Consider a random variable X with the following PMF:

$$p_X(k) = \frac{6}{\pi^2 k^2}, \qquad k = 1, 2, \ldots.$$

Absolutely summable

Definition

A discrete random variable X is **absolutely summable** if

$$\mathbb{E}[|X|] \stackrel{\text{def}}{=} \sum_{x \in X(\Omega)} |x| \, p_X(x) < \infty.$$

Only those absolutely summable random variables have expectations!

Questions?