

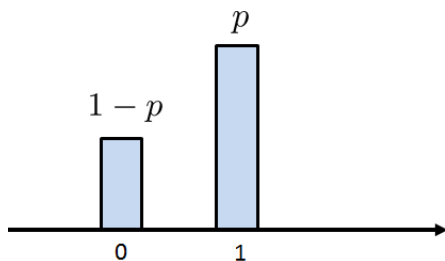
# ECE 302: Lecture 3.6 Bernoulli Random Variables

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## The random variable of flipping a coin



- How can this ever be interesting?
- It can be used to model randomized algorithm.
- It can be used to model social networks.

# Outline

- 3.1 Random variables
- 3.2 Probability mass functions (PMF)
- 3.3 Cumulative distribution functions (discrete case)
- 3.4 Expectation
- 3.5 Moments and variance
- 3.6 Bernoulli random variables
  - Definition of a Bernoulli random variable
  - Expectation, Variance
  - Application 1: Modeling randomized algorithms
  - Application 2: Modeling social networks
- 3.7 Binomial random variables
- 3.8 Geometric random variables
- 3.9 Poisson random variables

# Bernoulli Random Variable

## Definition

Let  $X$  be a **Bernoulli** random variable. Then, the PMF of  $X$  is

$$p_X(0) = 1 - p, \quad p_X(1) = p,$$

where  $0 < p < 1$  is called the Bernoulli parameter. We write

$$X \sim \text{Bernoulli}(p)$$

to say that  $X$  is drawn from a Bernoulli distribution with a parameter  $p$ .

**Example.** Coin flip.

# Mean, Variance

## Proposition

If  $X \sim \text{Bernoulli}(p)$ , then

$$\mathbb{E}[X] = p, \quad \mathbb{E}[X^2] = p, \quad \text{Var}[X] = p(1 - p).$$

## Analysis of the variance

When will the variance of a Bernoulli random variable be maximized?

- When  $p = 0$ ?
- When  $p = 1$ ?
- When  $p = \frac{1}{2}$ ?

## Rademacher random variable

- Bernoulli: 0 and 1
- Rademacher: +1 and -1

PMF of Rademacher is

$$p_X(-1) = \frac{1}{2}, \quad \text{and} \quad p_X(+1) = \frac{1}{2}.$$

You can show that following result:

### Proposition

*If  $X$  is a Rademacher random variable, then  $(X + 1)/2 \sim \text{Bernoulli}(1/2)$ .  
Conversely if  $Y \sim \text{Bernoulli}(1/2)$  then  $2Y - 1$  is Rademacher.*

## Application 1: Modeling randomized algorithms

Consider a large linear system:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

where  $M$  and  $N$  are very large.

### Example.

- Large network analysis (million-node network)
- Large-scale inverse problem (giga-pixel deconvolution)
- Large-scale routing (air traffic control)
- Large-scale decomposition (genome analysis)



## Application 1: Modeling randomized algorithms

**Brute-force computation:**

$$y_i = \sum_{j=1}^N a_{ij}x_j.$$

**Randomized computation:** Let  $I_j \sim \text{Bernoulli}(p_j)$ . Then,

$$\hat{y}_i = \sum_{j=1}^N a_{ij}x_j I_j / p_j,$$

- If  $I_j = 0$ , then you can skip that entry.

## Application 1: Modeling randomized algorithms

- The compensation by  $p_j$  is to ensure  $\mathbb{E}[\hat{y}_i] = y_i$ . How?

We can prove that with extremely high probability, the deviation between  $\hat{y}_i$  and  $y_i$  is very small.

Drineas, Kannan, Mahoney, Fast Monte Carlo algorithms for matrices, SIAM Journal on Computing 36(1), pp.132-157, 2006.

## Application 2: Modeling social networks

### Erdos-Renyi Graph

- A very famous (and simple) model for large networks.
- Is used nowadays to study network structures, e.g. Facebook, Google.

**Graph**

**Matrix**

## Application 2: Modeling social networks

Erdos-Renyi Graph says:

$$A_{ij} \sim \text{Bernoulli}(p), \quad (1)$$

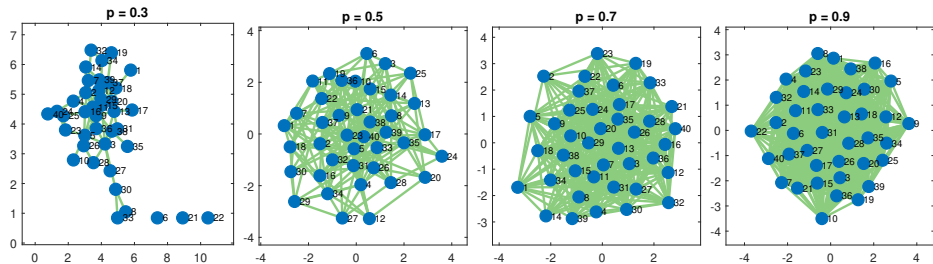


Figure: Erdős-Rényi graph with different parameters  $p = 0.3, 0.5, 0.7, 0.9$ .

**Questions?**