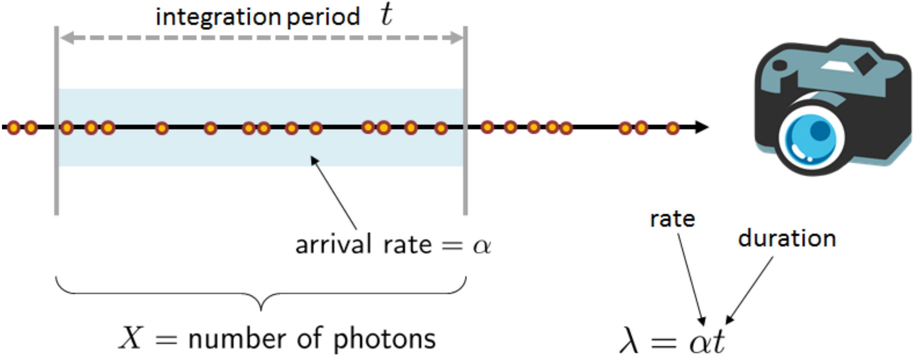
ECE 302: Lecture 3.9 Poisson Random Variables

Prof Stanley Chan

School of Electrical and Computer Engineering
Purdue University

Motivating Story: Cameras



$$\mathbb{P}[X = k] = \frac{\lambda^k}{k!} e^{-\lambda}$$

Mysteries about Poisson Random Variables

If you have learned Poisson random variables before,

- Your teacher probably asked you to memorize the PMF:

$$p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \qquad k = 1, 2, \ldots.$$

- And then you will learn how to compute the mean and variance.
- Have you ever wondered why the Poisson PMF is defined in that way?
- Is there a principled way of deriving this formula?
- Besides the formula, what else can Poisson buy you?
- Why are Poisson random variables so important for computer vision, especially low-light imaging?
- We are going to tell you all these in today's lecture.

Outline

- 3.1 Random variables
- 3.2 Probability mass functions (PMF)
- 3.3 Cumulative distribution functions (discrete case)
- 3.4 Expectation
- 3.5 Moments and variance
- 3.6 Bernoulli random variables
- 3.7 Binomial random variables
- 3.8 Geometric random variables
- 3.9 Poisson random variables
    - Definition of Poisson
    - Demystifying the mean and variance
    - Origin of Poisson
    - Poisson approximation to Binomial

Definition of Poisson Random Variable

Definition

Let X be a **Poisson** random variable. Then, the PMF of X is

$$p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \qquad k = 1, 2, \ldots,$$

where $\lambda > 0$ is the Poisson rate. We write

$$X \sim \text{Poisson}(\lambda)$$

to say that X is drawn from a Poisson distribution with a parameter λ.

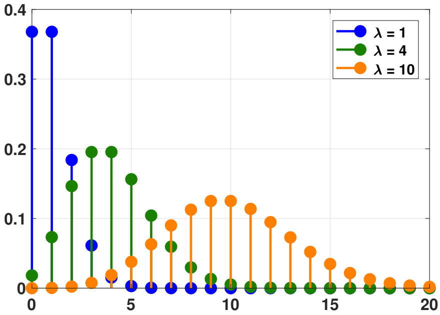
Understanding the parameter:

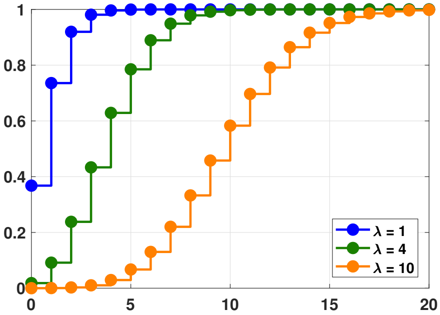
- X = number of arrivals
- α = arrival rate = number per unit time
- t = time
- So, $\lambda = \alpha t$ = average number within t unit time.

Shape of the Poisson PMF and CDF

The CDF of Poisson is

$$F_X(k) = \mathbb{P}[X \leq k] = \sum_{\ell=0}^{k} \frac{\lambda^\ell}{\ell!} e^{-\lambda}. \tag{1}$$





Example

Example 1. In a telephone call center, the average number of phone calls is λ calls. Let X be the actual number of calls. Find $\mathbb{P}[X > 4]$ and $\mathbb{P}[X \le 5]$.

Solution. By using a Poisson random variable with parameter λ, we can show that the probabilities are

$$\mathbb{P}[X > 4] = 1 - \mathbb{P}[X \le 4] = 1 - \sum_{k=0}^{4} \frac{\lambda^k}{k!} e^{-\lambda},$$

$$\mathbb{P}[X \le 5] = \sum_{k=0}^{5} \frac{\lambda^k}{k!} e^{-\lambda}.$$

Remark:

- $\lambda =$ average number of calls
- E.g., $\lambda = 1$ can be achieved by $\alpha = 0.5$ and $t = 2$, or $\alpha = 3$ and $t = 0.33$.

Outline

Mean and Variance

Proposition

If $X \sim \mathrm{Poisson}(\lambda)$, *then*

$$\mathbb{E}[X] = \lambda, \qquad \mathbb{E}[X^2] = \lambda + \lambda^2, \qquad \mathrm{Var}[X] = \lambda.$$

Interpreting the Mean and Variance

Proposition

If $X \sim \text{Poisson}(\lambda)$, *then*

$$\mathbb{E}[X] = \lambda, \qquad \mathbb{E}[X^2] = \lambda + \lambda^2, \qquad \text{Var}[X] = \lambda.$$

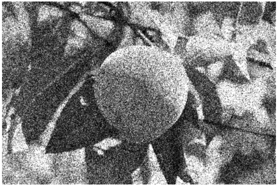
How do we understand the mean and the variance?

- Mean and variance are both λ
- Larger mean means higher variance!
- Brighter pixel means more noise!?

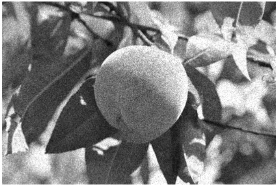
Wait a minute ...

This is what we get!

(a) $\lambda \in [0, 1]$



(b) $\lambda \in [0, 10]$



(c) $\lambda \in [0, 100]$

Inconsistent with "brighter pixel means more noise".

Interpreting the Mean and Variance

Let $\lambda =$ image intensity of one pixel, and let $X \sim \text{Poisson}(\lambda)$.

- It is true that "brighter pixel means more noise".

But this is not what you care! What you care is the **signal-to-noise ratio** (SNR).

$$\begin{aligned} \text{SNR} &= \frac{\mathbb{E}[X]}{\sqrt{\mathrm{Var}[X]}} \\ &= \frac{\lambda}{\sqrt{\lambda}} \\ &= \sqrt{\lambda}. \end{aligned}$$

Brighter pixel means more noise, but your signal is also stronger. Gain in signal overwhelms the gain in noise. So still a good deal!

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The origin of Poisson random variable

Question:

- We knew how to derive Bernoulli, Binomial and Geometric. How about Poisson?
- Where does the Poisson formula come from?

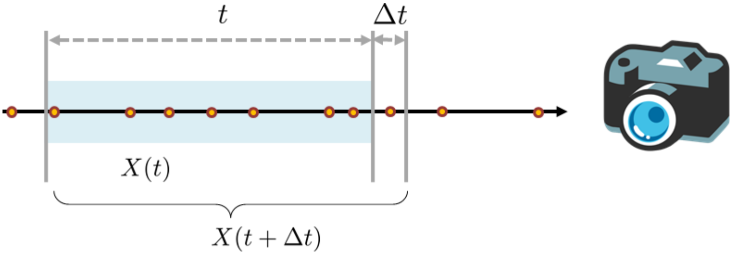
Why is this question so important?

- It tells your the intuition of Poisson.
- It helps you to formulate your next engineering problem.
- A lot of textbooks ignore this. A big mistake.

Our approach:

- Use our camera story
- Argument based on physics
- Very intuitive
- No jargon, no bubble

Elementary Assumptions



- Let Δt be very small. Assume

$$\mathbb{P}[X(t+\Delta t) - X(t) = 1] = \alpha \Delta t.$$

- Let Δt be very small. Assume no more than 1 electron arrives

$$\mathbb{P}[X(t+\Delta t) - X(t) = 0] = 1 - \alpha \Delta t.$$

- The number of impulses in non-overlapping time intervals are **independent**.

Let us zoom in to Δt

$$
\begin{aligned}
&\mathbb{P}[X(t+\Delta t)=k] \\
&= \mathbb{P}[X(t)=k-1]\cdot \underbrace{\mathbb{P}[\text{there is one photon in } \Delta t]}_{=\alpha\Delta t} \\
&\qquad\qquad + \\
&\qquad \mathbb{P}[X(t)=k]\cdot \underbrace{\mathbb{P}[\text{there is no photon in } \Delta t]}_{=1-\alpha\Delta t} \\
&= \mathbb{P}[X(t)=k-1]\cdot(\alpha\Delta t) + \mathbb{P}[X(t)=k]\cdot(1-\alpha\Delta t) \\
&= \mathbb{P}[X(t)=k] - \mathbb{P}[X(t)=k]\alpha\Delta t + \mathbb{P}[X(t)=k-1]\alpha\Delta t.
\end{aligned}
$$

This will give us

$$
\frac{\mathbb{P}[X(t+\Delta t)=k]-\mathbb{P}[X(t)=k]}{\Delta t} = \alpha\Big(\mathbb{P}[X(t)=k-1]-\mathbb{P}[X(t)=k]\Big).
$$

Differential equation!

Setting the limiting of $\Delta t \to 0$, we arrive at an ordinary differential equation

$$\frac{d}{dt}\mathbb{P}[X(t)=k] = \alpha\Big(\mathbb{P}[X(t)=k-1] - \mathbb{P}[X(t)=k]\Big). \tag{2}$$

Solution to this differential equation is

$$\mathbb{P}[X(t)=k] = \frac{(\alpha t)^k}{k!}e^{-\alpha t}$$

Why?

$$\begin{aligned}\frac{d}{dt}\mathbb{P}[X(t)=k] = \frac{d}{dt}\left(\frac{(\alpha t)^k}{k!}e^{-\alpha t}\right) &= \alpha k\frac{(\alpha t)^{k-1}}{k!}e^{-\alpha t} + (-\alpha)\frac{(\alpha t)^k}{k!}e^{-\alpha t} \\ &= \alpha\frac{(\alpha t)^{k-1}}{k!}e^{-\alpha t} - \alpha\frac{(\alpha t)^k}{k!}e^{-\alpha t} \\ &= \alpha\Big(\mathbb{P}[X(t)=k-1] - \mathbb{P}[X(t)=k]\Big),\end{aligned}$$

We found it

So we have found the PMF:

$$\mathbb{P}[X(t) = k] = \frac{(\alpha t)^k}{k!} e^{-\alpha t} \tag{3}$$

- α = rate = number per unit time
- t = time
- $\lambda = \alpha t$ = average number

Replacing $\lambda = \alpha t$, we obtain

$$\mathbb{P}[X = k] = \frac{\lambda^k}{k!} e^{-\lambda}$$

Outline

Poisson approximation to Binomial

Another very important result:

- When N is large, binomial is approximately Poisson

Another way to derive the Poisson formula:

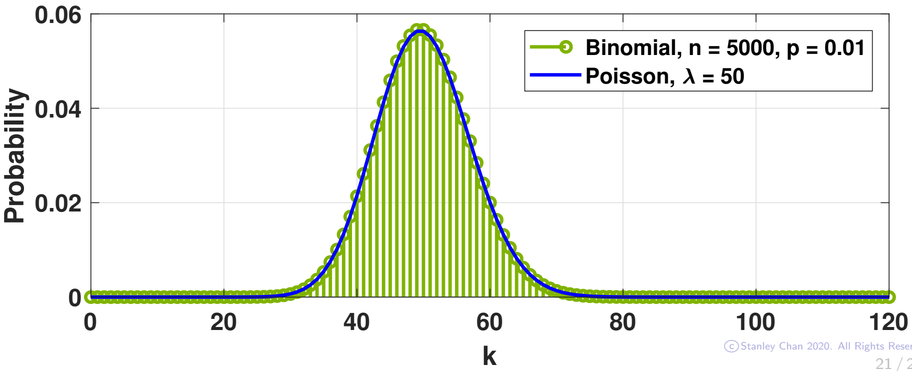
- Pure algebraic derivation
- Also correct
- Also easy to show
- Although the physics-based approach offers more intuition

Proposition

Poisson Approximation to Binomial. *For small p and large n,*

$$\binom{n}{k} p^k (1-p)^{n-k} \approx \frac{\lambda^k}{k!} e^{-\lambda},$$

where $\lambda \overset{def}{=} np$.



Example

Problem.

- Data arrival rate: $n = 10^9$ bits per second.
- Probability of having one error bit: $p = 10^{-9}$.
- In one second, how likely will we get $k = 5$ error bits?

Solution.

If you stick to binomial:

- Binomial: Flip coin 10^9 times. Get 5 heads.
- $\binom{10^9}{5}(10^{-9})^5(1-10^{-9})^{10^9-5}$.
- Numerical problem!!

If we use Poisson approximation:

- $\lambda = np = (10^9)(10^{-9}) = 1$.
- $\frac{1^5}{5!}e^{-1}$.
- Much easier!

Proof.

Let $\lambda = np$. Then,

$$
\begin{aligned}
&\binom{n}{k} p^k (1-p)^{n-k} \\
&= \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!} \frac{n(n-1)\dots(n-k+1)}{n \cdot n \cdots n} \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!} \underbrace{(1)\left(1 - \frac{1}{n}\right)\dots\left(1 - \frac{k-1}{n}\right)}_{\to 1 \text{ as } n \to \infty} \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-k}}_{\to 1 \text{ as } n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n \\
&= \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n .
\end{aligned}
$$

Proof.

We claim that $\left(1 - \frac{\lambda}{n}\right)^n \to e^{-\lambda}$. This can be proved by noting that

$$\log(1 + x) \approx x, \qquad x \ll 1.$$

It then follows that $\log\left(1 - \frac{\lambda}{n}\right) \approx -\frac{\lambda}{n}$. Hence, $\left(1 - \frac{\lambda}{n}\right)^n \approx e^{-\lambda}$

Questions?