ECE 302: Lecture 4.2 Expectation

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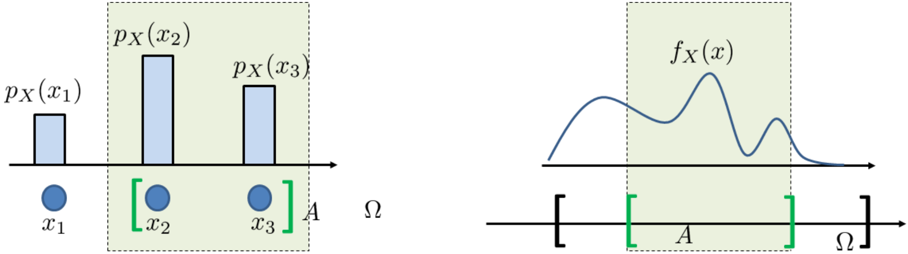
From last lecture



Today's lecture: How to compute expectation?

Outline

- Expectation
- Properties
- Existence of expectation
- Moment and variance

Definition

Definition

The expectation of a continuous random variable X is

$$\mathbb{E}[X] = \int_{\Omega} x \, f_X(x) dx. \tag{1}$$

Examples

Example 1. (Uniform random variable) Let X be a continuous random variable with PDF $f_X(x) = \frac{1}{b-a}$ for $a \leq x \leq b$, and 0 otherwise. Find $\mathbb{E}[X]$

Example 2. (Exponential random variable) Let X be a continuous random variable with PDF $f_X(x) = \lambda e^{-\lambda x}$, for $x \geq 0$. Find $\mathbb{E}[X]$.

Function of random variables

Theorem

Let $g : \Omega \to \mathbb{R}$ *be a function and* X *be a continuous random variable, then*

$$\mathbb{E}[g(X)] = \int_{\Omega} g(x)\, f_X(x) dx. \tag{2}$$

Example

Example 1. (Uniform random variable) Let X be a continuous random variable with $f_X(x) = \frac{1}{b-a}$ for $a \leq x \leq b$, and 0 otherwise. Find $\mathbb{E}[X^2]$.

Example

Example 2. Let Θ be a continuous random variable with PDF $f_\Theta(\theta) = \frac{1}{2\pi}$ for $0 \leq \theta \leq 2\pi$ and is 0 otherwise. Let $Y = \cos(\omega t + \Theta)$. Find $\mathbb{E}[Y]$.

Example

Example 3. Let $A \subseteq \Omega$. Let $\mathbb{I}_A(X)$ be an indicator function such that

$$\mathbb{I}_A(X) = \begin{cases} 1, & \text{if} \quad X \in A, \\ 0, & \text{if} \quad X \notin A. \end{cases}$$

Find $\mathbb{E}[\mathbb{I}_A(X)]$.

Properties

Theorem

- $\mathbb{E}[aX] = a\mathbb{E}[X]$: *Scalar multiple of random variable will scale the expectation.*
- $\mathbb{E}[X + a] = \mathbb{E}[X] + a$: *Constant addition of a random variable will offset the expectation.*
- $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$: *Linear transformation of a random variable will translate to the expectation.*

Existence of expectation

Definition

A random variable X has an expectation if it is **absolutely integrable**, i.e.,

$$\mathbb{E}[|X|] = \int_{\Omega} |x| f_X(x) dx < \infty. \tag{3}$$

Example. Let X be a Cauchy random variable with PDF

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \qquad x \in \mathbb{R}. \tag{4}$$

Then X does not have $\mathbb{E}[X]$.

Mean of Cauchy distribution

We can evaluate:

$$\begin{aligned}\mathbb{E}[|X|] &= \int_{-\infty}^{\infty} |x| \cdot \frac{1}{\pi(1+x^2)} dx \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{x}{(1+x^2)} dx.\end{aligned}$$

The integral gives

$$\int_0^{\infty} \frac{x}{(1+x^2)} dx = \frac{1}{2} \log(1+x^2) \Big|_0^{\infty} = \infty.$$

Moment and variance

Definition

The kth moment of a continuous random variables X is

$$\mathbb{E}[X^k] = \int_\Omega x^k f_X(x)dx. \tag{5}$$

Definition

The variance of a continuous random variables X is

$$\mathrm{Var}[X] = \mathbb{E}[(X-\mu)^2] = \int_\Omega (x-\mu)^2 f_X(x)dx, \tag{6}$$

where $\mu \stackrel{\text{def}}{=} \mathbb{E}[X]$.

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mu^2, \tag{7}$$

Example

Example. (Uniform random variable) Let X be a continuous random variable with PDF $f_X(x) = \frac{1}{b-a}$ for $a \leq x \leq b$, and 0 otherwise. Find $\mathrm{Var}[X]$.

Hint: We have shown that $\mathbb{E}[X] = \frac{a+b}{2}$ and $\mathbb{E}[X^2] = \frac{a^2+ab+b^2}{3}$.

Summary

- Expectation
  - $\mathbb{E}[X] = \int_\Omega x f_X(x) dx$
  - $\mathbb{E}[g(X)] = \int_\Omega g(x) f_X(x) dx$
- Properties
  - $\mathbb{E}[aX] = a\mathbb{E}[X]$
  - $\mathbb{E}[X + a] = \mathbb{E}[X] + a$
  - $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$
- Existence of expectation
  - $\mathbb{E}[X]$ exists when $\mathbb{E}[|X|] < \infty$
- Moment and variance
  - $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mu^2$

Questions?