ECE 302: Lecture 4.3 Cumulative Distribution Function

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Outline

Cumulative distribution function (CDF):

$$F_X(x) \stackrel{\text{def}}{=} \mathbb{P}[X \leq x] \tag{1}$$

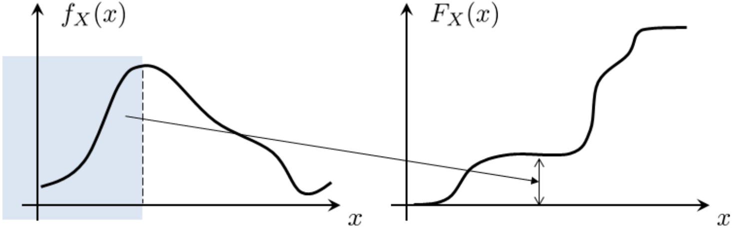
- What is a CDF?
- What are the properties of CDF?
- How are CDFs related to PDF?

Definition

Definition

Let X be a continuous random variable with a sample space $\Omega = \mathbb{R}$. The **cumulative distribution function (CDF)** of X is

$$F_X(x) \stackrel{\text{def}}{=} \mathbb{P}[X \leq x]. \tag{2}$$

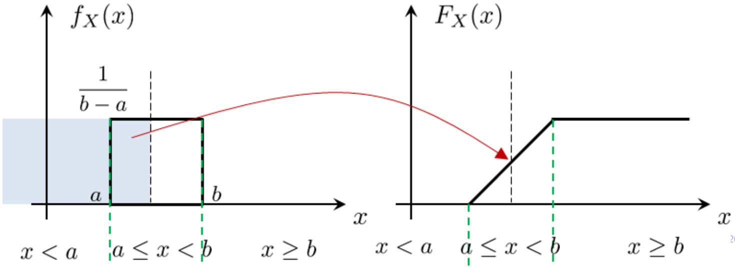


Example

Question. (Uniform random variable) Let X be a continuous random variable with PDF $f_X(x) = \frac{1}{b-a}$ for $a \leq x \leq b$, and is 0 otherwise. Find the CDF of X.

Solution.

$$F_X(x) = \qquad\qquad = \begin{cases} 0, & x \leq a, \\ \frac{x-a}{b-a}, & a < x \leq b, \\ 1, & x > b. \end{cases}$$

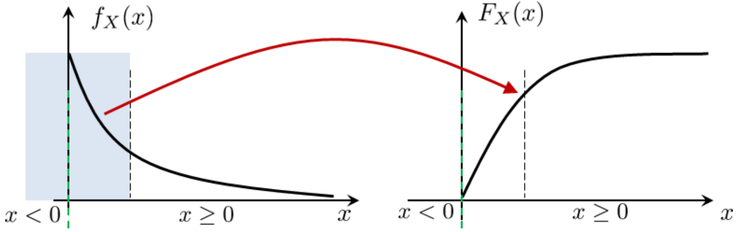


Example 2

Question.(Exponential random variable) Let X be a continuous random variable with PDF $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$, and is 0 otherwise. Find the CDF of X.

Solution.

$$F_X(x) = \qquad = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$



Properties 1-3

Theorem

Let X be a random variable (either continuous or discrete), then the CDF of X has the following properties:

(i) *The CDF is a* **non-decreasing**.

(ii) *The* **maximum** *of the CDF is when $x = \infty$: $F_X(+\infty) = 1$.*

(iii) *The* **minimum** *of the CDF is when $x = -\infty$: $F_X(-\infty) = 0$.*

Property 4

Theorem

Let X be a continuous random variable. If the CDF F_X is continuous at any $a \leq x \leq b$, then

$$\mathbb{P}[a \leq X \leq b] = F_X(b) - F_X(a). \tag{3}$$

Example

Example 1. (Exponential random variable.) $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$, $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. Find $\mathbb{P}[1 \leq X \leq 3]$.

(a) PDF approach:

$$\mathbb{P}[1 \leq X \leq 3] = \qquad = e^{-3\lambda} - e^{-\lambda}$$

(b) CDF approach:

$$\mathbb{P}[1 \leq X \leq 3] = \qquad = e^{-3\lambda} - e^{-\lambda}$$

Example

Example 2. Let X be a random variable with PDF $f_X(x) = 2x$ for $0 \leq x \leq 1$, and is 0 otherwise.

(a) Find CDF.

$$F_X(x) = \qquad\qquad = x^2, \qquad 0 \leq x \leq 1.$$

(b) Find $\mathbb{P}[1/3 \leq X \leq 1/2]$.

$$\mathbb{P}\left[\frac{1}{3} \leq X \leq \frac{1}{2}\right] = \qquad\qquad = \frac{5}{36}. \qquad \square$$

Left and Right Continuous

Definition

A function $F_X(x)$ is said to be

- **Left-continuous** at $x = b$ if $F_X(b) = F_X(b^-) \stackrel{\text{def}}{=} \lim_{h \to 0} F_X(b - h)$;
- **Right-continuous** at $x = b$ if $F_X(b) = F_X(b^+) \stackrel{\text{def}}{=} \lim_{h \to 0} F_X(b + h)$;
- **Continuous** at $x = b$ if it is both right-continuous and left-continuous at $x = b$. In this case, we have

$$\lim_{h \to 0} F_X(b - h) = \lim_{h \to 0} F_X(b + h) = F(b).$$

Left and Right Continuous

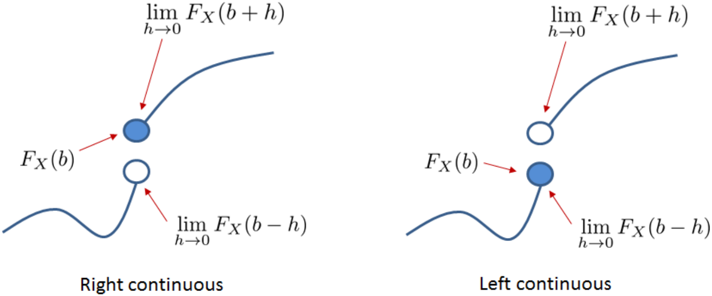


Figure: The definition of left and right continuous at a point b.

Property 5: CDF must be right continuous

Theorem

For any random variable X (discrete or continuous), $F_X(x)$ is always **right-continuous**. *That is,*

$$F_X(b) = F_X(b^+) \overset{def}{=} \lim_{h \to 0} F_X(b+h) \tag{4}$$

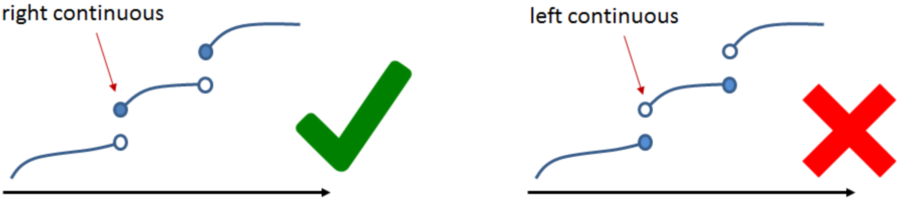


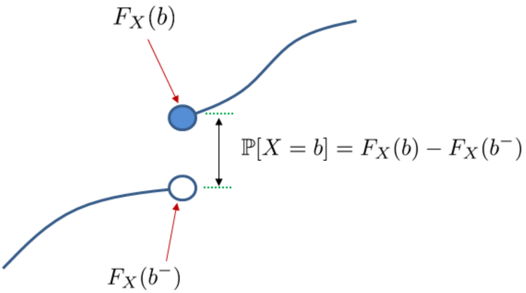
Figure: A CDF must be right continuous

Property 6: Jump

Theorem

For any random variable X (discrete or continuous), $\mathbb{P}[X = b]$ is

$$\mathbb{P}[X = b] = \begin{cases} F_X(b) - F_X(b^-), & \text{if } F_X \text{ is discontinuous at } x = b \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$



Example

Example. Consider a random variable X with a PDF

$$f_X(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ \frac{1}{2}, & x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

Find CDF.

(a) $0 \leq x < 1$:

$$F_X(x) = \qquad\qquad = \frac{x^2}{2}, \quad 0 \leq x < 1.$$

(b) $1 \leq x < 3$:

$$F_X(x) = \qquad\qquad = \frac{1}{2}, \qquad 1 \leq x < 3.$$

Example

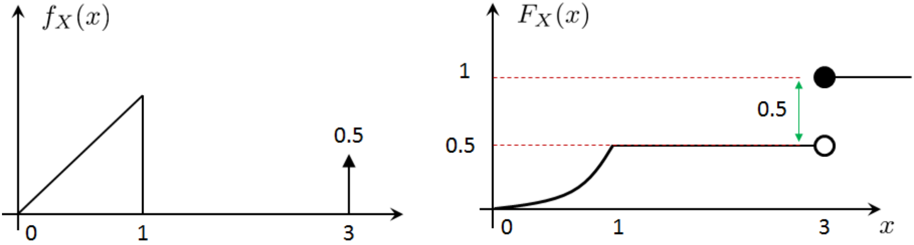


Figure: An example of converting a PDF to a CDF.

Example

(c) $x = 3$:

$$F_X(3) = \qquad\qquad = 1, \qquad x = 3.$$

(d) $x > 3$:

$$F_X(x) = \qquad\qquad = 1, \qquad x > 3.$$

Therefore,

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{x^2}{2}, & 0 \le x < 1, \\ \frac{1}{2}, & 1 \le x < 3, \\ 1, & x \ge 3. \end{cases}$$

Retrieving PDF from CDF

Theorem

The **probability density function** *(PDF) is the derivative of the cumulative distribution function (CDF):*

$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{d}{dx} \int_{-\infty}^{x} f_X(x')dx', \tag{6}$$

provided F_X is differentiable at x. If F_X is not differentiable at x, then,

$$f_X(x) = \mathbb{P}[X = x] = F_X(x) - \lim_{h \to 0} F_X(x - h). \tag{7}$$

Example

Consider a CDF

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - \frac{1}{4}e^{-2x}, & x \geq 0. \end{cases}$$

Find PDF $f_X(x)$.

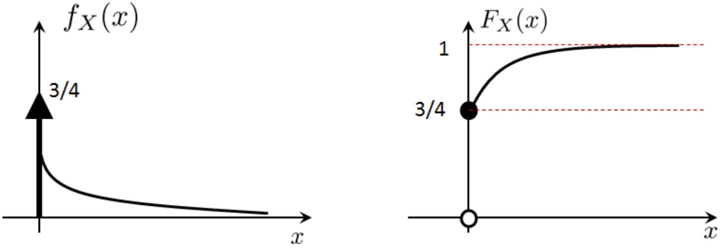


Figure: An example of converting a PDF to a CDF.

Example

(a) When $x < 0$:

$$f_X(x) = \qquad = 0$$

(b) When $x = 0$:

$$f_X(x) = \qquad = \frac{3}{4}$$

(c) When $x > 0$:

$$f_X(x) = \qquad = \frac{1}{2}e^{-2x}$$

Therefore, the overall PDF is

$$f_X(x) = \begin{cases} 0, & x < 0, \\ \frac{3}{4}, & x = 0, \\ \frac{1}{2}e^{-2x}, & x > 0. \end{cases}$$

Summary

The **cumulative distribution function (CDF)** of X is

$$F_X(x) \overset{\text{def}}{=} \mathbb{P}[X \leq x]$$

CDF must satisfy these **properties**:

- Non-decreasing, $F_X(-\infty) = 0$, and $F_X(\infty) = 1$.
- $\mathbb{P}[a \leq X \leq b] = F_X(b) - F_X(a)$.
- Right continuous: Solid dot on at the start.
- If discontinuous at b, then $\mathbb{P}[X = b] = \text{Gap}$.

Relationship between CDF and PDF:

- PDF $\rightarrow$ CDF: Integration
- CDF $\rightarrow$ PDF: Differentiation

Questions?