ECE 302: Lecture 4.6 Exponential Random Variable

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Energy Efficient Escalator

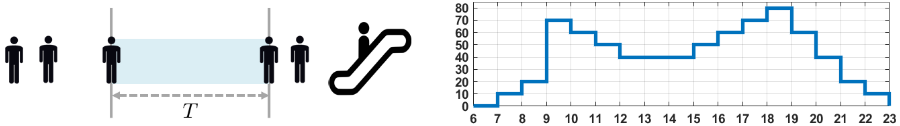


Figure: The variable-speed escalator problem. [Left] We model the passengers as independent Poisson arrivals. Thus, the inter-arrival time is exponential. [Right] A hypothetical passenger arrival rate (number of people per minute), from 06:00 to 23:00.

Question:

- Two modes: High-speed mode, and low-speed mode.
- If no passenger arrives in more than τ seconds, then switch to low-speed mode.
- On average, how much saving will you get?

Outline

Overall schedule:

- Continuous random variables, PDF
- CDF
- Expectation
- Mean, mode, median
- Common random variables
  - Uniform
  - Exponential
  - Gaussian
- Transformation of random variables
- How to generate random numbers

Today's lecture:

- Definition of exponential random variables
- Properties
- Origin
- Example: Energy efficient escalator

Definition

Definition

Let X be an exponential random variable. The PDF of X is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $\lambda > 0$ is a parameter. We write

$$X \sim \text{Exponential}(\lambda)$$

to say that X is drawn from an exponential distribution of parameter λ.

PDF

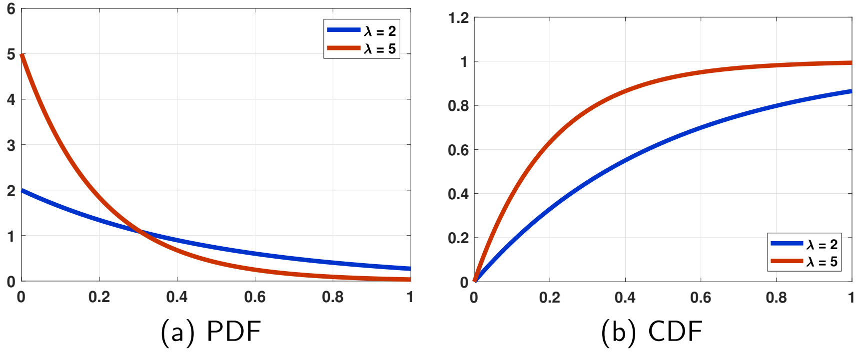


Figure: The PDF and CDF of $X \sim \text{Exponential}(\lambda)$.

Note that the initial value $f_X(0)$ is

$$f_X(0) = \lambda e^{-\lambda 0} = \lambda.$$

Therefore, as long as $\lambda > 1$, $f_X(0)$ will exceed 1.

CDF

The CDF of an exponential random variables can be determined by

$$
\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(t)dt \\
&= \int_{0}^{x} \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}, \qquad\qquad x \geq 0.
\end{aligned}
$$

Therefore, if we consider the entire real line, then the CDF is

$$
F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases} \tag{2}
$$

Mean and Variance

Theorem

If $X \sim \mathrm{Exponential}(\lambda)$, *then*

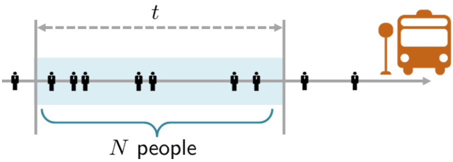
$$\mathbb{E}[X] = \frac{1}{\lambda}, \quad \textit{and} \quad \mathrm{Var}[X] = \frac{1}{\lambda^2}.$$

$$\begin{aligned}
\mathbb{E}[X] &= \\
&= \qquad = \frac{1}{\lambda} \\
\mathbb{E}[X^2] &= \\
&= \qquad = \frac{2}{\lambda^2}.
\end{aligned}$$

Origin of Exponential Random Variables

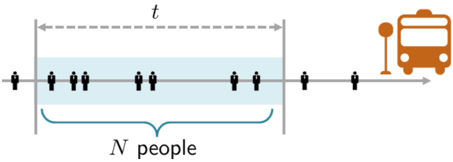
What is the origin of exponential random variables?

- An exponential random variable is the **inter-arrival time** between two consecutive Poisson events.
- That is, how much time it takes to go from N Poisson counts to $N+1$ Poisson counts.



Question: Find the inter-arrival time between two people.

Deriving Exponential from Scratch



- Imagine that you are waiting a bus.
- The people come with an arrival rate λ per unit time.
- Thus, for a time period of t, the average number of people that arrive is λt.
- Let N be a random variable denoting the number of people. We assume that N is Poisson with a parameter λt.
- That is, for any duration t, the probability of observing n people follows the PMF

$$\mathbb{P}[N = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

Inter-arrival Time T

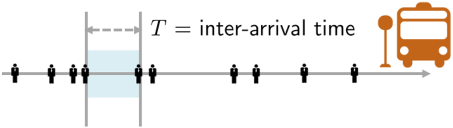


Figure: The inter-arrival time T between two consecutive Poisson events is an exponential random variable.

$$
\begin{aligned}
\mathbb{P}[T > t] &\overset{(a)}{=} \mathbb{P}[\text{inter-arrival time } > t] \\
&\overset{(b)}{=} \mathbb{P}[\text{no arrival in } t] \\
&\overset{(c)}{=} \mathbb{P}[N = 0] \\
&= \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t}.
\end{aligned}
$$

CDF and PDF of Inter-arrival Time

Since $\mathbb{P}[T > t] = 1 - F_T(t)$, where $F_T(t)$ is the CDF of T, we can show that

$$F_T(t) = 1 - e^{-\lambda t}$$
$$f_T(t) = \frac{d}{dt} F_T(t) = \lambda e^{-\lambda t}.$$

Therefore, the inter-arrival time T follows an **exponential distribution**.

Question: When to use exponential?

- It is the random variable for *time* — inter-arrival time.
- It is derived from Poisson.
- We use it to model photon arrival time, passenger arrival time, etc.

Energy Efficient Escalator

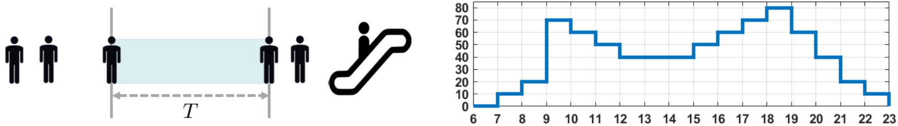


Figure: The variable-speed escalator problem. [Left] We model the passengers as independent Poisson arrivals. Thus, the inter-arrival time is exponential. [Right] A hypothetical passenger arrival rate (number of people per minute), from 06:00 to 23:00.

Question:

- Two modes: High-speed mode, and low-speed mode.
- If no passenger arrives in more than τ seconds, then switch to low-speed mode.
- On average, how much saving will you get?

Setting up the problem

- The people come with an arrival rate λ per unit time.
- Thus, for a time period of t, the average number of people that arrive is λt.
- Let N be a random variable denoting the number of people. We assume that N is Poisson with a parameter λt.
- That is, for any duration t, the probability of observing n people follows the PMF

$$\mathbb{P}[N = n] =$$

- Let T be the inter-arrival time. Then,

$$\begin{aligned}\mathbb{P}[T > t] &= \\ &= \\ &= \qquad\qquad = e^{-\lambda t}.\end{aligned}$$

Setting up the problem

- So,

$$F_T(t) = $$
$$f_T(t) = \qquad = \lambda e^{-\lambda t}.$$

- and then, ... let Y be

$$Y = \begin{cases} \quad , & T > \tau, \\ \quad , & T \leq \tau. \end{cases}$$

- We need to find the PDF of Y, and then find $\mathbb{E}[Y]$.

Finding the PDF of Y

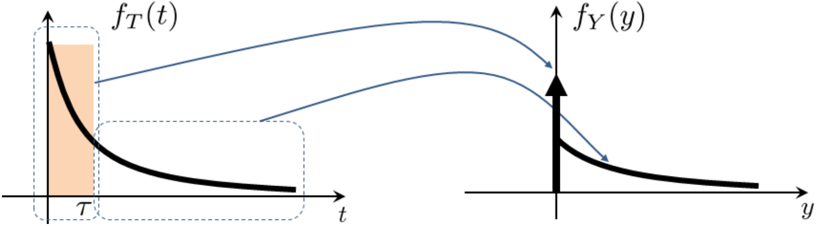


Figure: The escalator problem requires modeling the cutoff threshold τ such that if $T > \tau$, the saving is $Y = T - \tau$. If $T < \tau$, then $Y = 0$. The left hand side of the figure shows how the PDF of Y is constructed.

Finding the PDF of Y

When $Y = 0$, there is a probability mass such that

$$
\begin{aligned}
f_Y(0) &= \\
&= \\
&= 1 - e^{-\lambda\tau}.
\end{aligned}
$$

For other values of y, we can show that

$$f_Y(y) = \qquad\qquad , \qquad y > 0.$$

Therefore,

$$f_Y(y) = \begin{cases} 1 - e^{-\lambda\tau}, & y = 0, \\ \lambda e^{-\lambda(y+\tau)}, & y > 0. \end{cases}$$

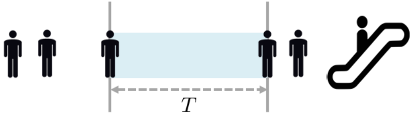
Average Saving

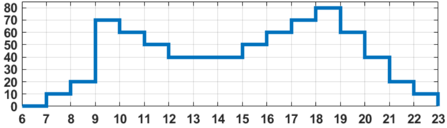
How much time for low-speed?

$$
\begin{aligned}
\mathbb{E}[Y] &= (0)\mathbb{P}[Y=0] + \int_0^\infty y f_Y(y)dy \\
&= 0 + \int_0^\infty y \cdot \; \lambda e^{-\lambda(y+\tau)} dy \\
&= \text{Exercise.}
\end{aligned}
$$

How much saving?

- If you need to spend \$b for low-speed mode per minute, then
- $b\mathbb{E}[Y]$ is the bill you need to pay for low-speed.
- Also need to vary your calculation based on the time period.





Summary

What are exponential random variables?

- It is the random variable for *time* — inter-arrival time.
- It is derived from Poisson.
- We use it to model photon arrival time, passenger arrival time, etc.
- Widely used in internet traffic, air traffic, congestion analysis.
- Also used in time-of-flight (depth sensing) cameras, LiDAR, etc.

When will it fail?

- If the underlying process deviates from the ideal Poisson.
- E.g., people are not coming independently.
- E.g., you model the entire 24 hour period using one λ.

Questions?