ECE 302: Lecture 5.4 Conditional Distributions

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Outline

- Joint PDF and CDF
- Joint Expectation
- Conditional Distribution
- Conditional Expectation
- Sum of Two Random Variables
- Random Vectors
- High-dimensional Gaussians and Transformation
- Principal Component Analysis

Today's lecture

- Joint PMF
- Joint PDF
- Decomposition – Law of total probability

Conditional PMF

Definition

Let X and Y be two discrete random variables. The **conditional PMF** of X given Y is

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}. \quad (1)$$

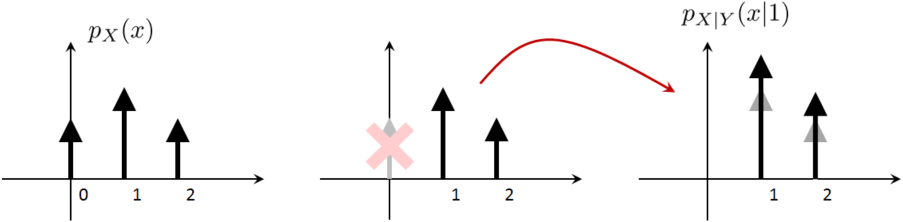

Figure: Suppose X is the sum of two coins, and Y is the first coin. When X is unconditioned, the PMF is just $p_X(x)$. When X is conditioned on $Y = 1$, then $X = 0$ cannot happen. Therefore, the resulting PMF $p_{X|Y}(x|1)$ only has two states. After normalization we obtain the conditional PMF.

Example 1

Consider a joint PMF given in the following table. Find the conditional PMF $p_{X|Y}(x|1)$ and the marginal PMF $p_X(x)$.

	Y=			
	1	2	3	4
X = 1	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{0}{20}$
2	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{1}{20}$
3	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{1}{20}$
4	$\frac{0}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$

Solution:

$$p_X(x) = \begin{bmatrix} \frac{3}{20} & \frac{6}{20} & \frac{8}{20} & \frac{3}{20} \end{bmatrix}$$

$$p_{X|Y}(x|1) = \frac{p_{X,Y}(x,1)}{p_Y(1)} = \frac{\begin{bmatrix} \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & \frac{0}{20} \end{bmatrix}}{\frac{3}{20}} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

Example 2

Consider two random variables X and Y defined as follows.

$$Y = \begin{cases} 10^2, & \text{with prob } 5/6, \\ 10^4, & \text{with prob } 1/6. \end{cases} \quad X = \begin{cases} 10^{-4}Y, & \text{with prob } 1/2, \\ 10^{-3}Y, & \text{with prob } 1/3, \\ 10^{-2}Y, & \text{with prob } 1/6. \end{cases}$$

Find $p_{X|Y}(x|y)$, $p_X(x)$ and $p_{X,Y}(x,y)$.

Solution.

$$p_{X|Y}(x|10^2) = \begin{cases} 1/2, & \text{if } x = 0.01, \\ 1/3, & \text{if } x = 0.1, \\ 1/6, & \text{if } x = 1. \end{cases}$$

$$p_{X|Y}(x|10^4) = \begin{cases} 1/2, & \text{if } x = 1, \\ 1/3, & \text{if } x = 10, \\ 1/6, & \text{if } x = 100. \end{cases}$$

Example 2

Therefore, the joint PMF $p_{X,Y}(x,y)$ is given by the following table.

10^4	0	0	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$
10^2	$\frac{5}{12}$	$\frac{5}{18}$	$\frac{5}{36}$	0	0
	0.01	0.1	1	10	100

The marginal PMF $p_X(x)$ is

$$p_X(x) = \sum_y p_{X,Y}(x,y) = \begin{bmatrix} \frac{5}{12} & \frac{5}{18} & \frac{2}{9} & \frac{1}{18} & \frac{1}{36} \end{bmatrix}.$$

Example 3

Let us follow up on Example 1. What is the probability that $\mathbb{P}[X > 2|Y = 1]$? What is the probability that $\mathbb{P}[X > 2]$?

Solution. Since the problem asks about the conditional probability, we know that it can be computed by using the conditional PMF. This gives us

$$\begin{aligned}\mathbb{P}[X > 2|Y = 1] &= \sum_{x>2} p_{X|Y}(x|1) \\ &= \cancel{p_{X|Y}(1|1)} + \cancel{p_{X|Y}(2|1)} + \underbrace{p_{X|Y}(3|1)}_{\frac{1}{3}} + \underbrace{p_{X|Y}(4|1)}_{0} = \frac{1}{3}.\end{aligned}$$

The other probability is

$$\mathbb{P}[X > 2] = \sum_{x>2} p_X(x) = \cancel{p_X(1)} + \cancel{p_X(2)} + \underbrace{p_X(3)}_{\frac{8}{20}} + \underbrace{p_X(4)}_{\frac{3}{20}} = \frac{11}{20}.$$

Decomposition

Theorem

Let X and Y be two discrete random variables, and let A be an event. Then

$$(i) \qquad \mathbb{P}[X \in A \,|\, Y = y] = \sum_{x \in A} p_{X|Y}(x|y)$$

$$(ii) \qquad \mathbb{P}[X \in A] = \sum_{x \in A} \sum_{y \in \Omega_Y} p_{X|Y}(x|y) p_Y(y) = \sum_{y \in \Omega_Y} \mathbb{P}[X \in A \,|\, Y = y] p_Y(y)$$

What is the rule of thumb for conditional distribution?

- The PMF/PDF should *match* with the probability you are finding.
- If you are finding conditional probability $\mathbb{P}[X \in A | Y = y]$, then use the conditional PMF $p_{X|Y}(x|y)$.
- If you are finding the probability $\mathbb{P}[X \in A]$, then use the marginal PMF $p_X(x)$.

Conditional PDF

Definition

Let X and Y be two continuous random variables. The **conditional PDF** of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}. \tag{2}$$

Theorem

Let X and Y be continuous random variables. Let A be an event. Then

$$(i) \qquad \mathbb{P}[X \in A \mid Y = y] = \int_A f_{X|Y}(x|y)dx$$

$$(ii) \qquad \mathbb{P}[X \in A] = \int_{\Omega_Y} \mathbb{P}[X \in A \mid Y = y] f_Y(y)dy.$$

Example 1

Example 1. Let X and Y:

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x}e^{-y}, & 0 \le y \le x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional PDF $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.
Solution.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^x 2e^{-x}e^{-y}dy = 2e^{-x}(1-e^{-x})$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_y^{\infty} 2e^{-x}e^{-y}dx = 2e^{-2y}.$$

Therefore, the conditional PDFs are

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2e^{-x}e^{-y}}{2e^{-2y}} = e^{-(x+y)}, \quad x \ge y$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2e^{-x}e^{-y}}{2e^{-x}(1-e^{-x})} = \frac{e^{-y}}{1-e^{-x}}, \quad 0 \le y < x.$$

Example 2

Example 2. This example considers a classical detection problem. Let X be a random bit such that

$$X = \begin{cases} +1, & \text{with prob } 1/2 \\ -1, & \text{with prob } 1/2. \end{cases}$$

Suppose that X is transmitted over a noisy channel so that the observed signal is

$$Y = X + N,$$

where $N \sim \text{Gaussian}(0, 1)$ is the noise which is independent to the signal X. Find the probability $\mathbb{P}[X = +1 \mid Y > 0]$ and $\mathbb{P}[X = -1 \mid Y > 0]$.

Example 2

Solution. First of all, we know that

$$f_{Y|X}(y|+1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}}, \qquad \text{and} \qquad f_{Y|X}(y|-1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+1)^2}{2}}.$$

Therefore, integrating y from 0 to ∞ will give us

$$\begin{aligned} \mathbb{P}[Y > 0 \,|\, X = +1] &= \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}} \, dy \\ &= 1 - \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}} \, dy \\ &= 1 - \Phi\left(\frac{0-1}{1}\right) = 1 - \Phi(-1). \end{aligned}$$

Similarly, we have $\mathbb{P}[Y > 0 \,|\, X = -1] = 1 - \Phi(+1)$.

By Bayes theorem,

$$\mathbb{P}[X = +1 \mid Y > 0] = \frac{\mathbb{P}[Y > 0 \mid X = +1]\mathbb{P}[X = +1]}{\mathbb{P}[Y > 0]}.$$

The denominator can be found by using the law of total probability:

$$\begin{aligned}\mathbb{P}[Y > 0] &= \mathbb{P}[Y > 0 \mid X = +1]\mathbb{P}[X = +1] + \mathbb{P}[Y > 0 \mid X = -1]\mathbb{P}[X = -1] \\ &= 1 - \frac{1}{2}\left(\Phi(+1) + \Phi(-1)\right) = \frac{1}{2},\end{aligned}$$

because $\Phi(+1) + \Phi(-1) = \Phi(+1) + 1 - \Phi(+1) = 1$. Therefore,

$$\mathbb{P}[X = +1 \mid Y > 0] = 1 - \Phi(-1) = 0.8413.$$

The implication is that if $Y > 0$, the probability $\mathbb{P}[X = +1 \mid Y > 0] = 0.8413$. The complement of this result gives that $\mathbb{P}[X = -1 \mid Y > 0] = 1 - 0.8413 = 0.1587$.

Summary

Definition

Let X and Y be two continuous random variables. The **conditional PDF** of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}. \tag{3}$$

Theorem

Let X and Y be continuous random variables. Let A be an event. Then

$$(i) \qquad \mathbb{P}[X \in A \mid Y = y] = \int_A f_{X|Y}(x|y)dx$$

$$(ii) \qquad \mathbb{P}[X \in A] = \int_{\Omega_Y} \mathbb{P}[X \in A \mid Y = y] f_Y(y)dy.$$

Questions?