ECE 302: Lecture 5.5 Conditional Expectation

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Conditional PMF and PDF

Definition

Let X and Y be two discrete random variables. The **conditional PMF** of X given Y is

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}. \tag{1}$$

Definition

Let X and Y be two continuous random variables. The **conditional PDF** of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}. \tag{2}$$

Outline

- Joint PDF and CDF
- Joint Expectation
- Conditional Distribution
- Conditional Expectation
- Sum of Two Random Variables
- Random Vectors
- High-dimensional Gaussians and Transformation
- Principal Component Analysis

Today's lecture

- What is conditional expectation
- Law of total expectation
- Examples

Conditional Expectation

Definition

The conditional expectation of X given $Y = y$ is

$$\mathbb{E}[X \mid Y = y] = \sum_{x} x p_{X|Y}(x|y) \tag{3}$$

for the discrete random variables, and

$$\mathbb{E}[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx \tag{4}$$

for the continuous random variables.

What is conditional expectation?

What is conditional expectation?

- $\mathbb{E}[X|Y=y]$ is the expectation using $f_{X|Y}(x|y)$.
- The integration is taken w.r.t. x, because $Y=y$ is given and fixed.

Law of Total Expectation

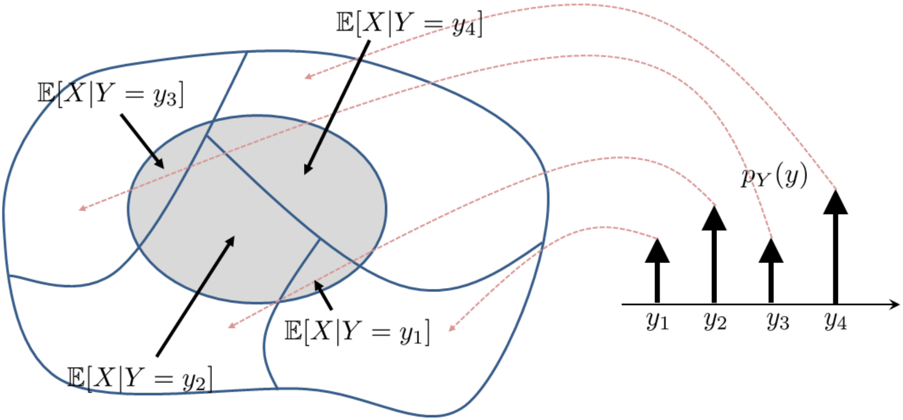
Theorem (Law of Total Expectation)

$$\mathbb{E}[X] = \sum_y \mathbb{E}[X|Y=y]p_Y(y), \quad or \quad \mathbb{E}[X] = \int_{-\infty}^{\infty} \mathbb{E}[X|Y=y]f_Y(y)dy. \tag{5}$$

What is law of total expectation?

- Law of total expectation is a **decomposition** rule.
- It decomposes $\mathbb{E}[X]$ into smaller/easier conditional expectations.

Law of Total Expectation



Proof

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

$$=$$

$$= \sum_x \sum_y x p_{X|Y}(x|y) p_Y(y)$$

$$=$$

$$= \sum_y \mathbb{E}[X|Y=y] p_Y(y). \qquad \square$$

$\mathbb{E}_{X|Y}[X|Y]$

$\mathbb{E}_{X|Y}[X|Y=y]=$ a deterministic function in y.

$\mathbb{E}_{X|Y}[X|Y]=$ a function of the random variable Y.

An alternative form

Some people prefer to write the above theorem in a more compact form.

Corollary

Let X and Y be two random variables. Then,

$$\mathbb{E}[X] = \mathbb{E}_Y\left[\mathbb{E}_{X|Y}[X|Y]\right]. \tag{6}$$

Proof.

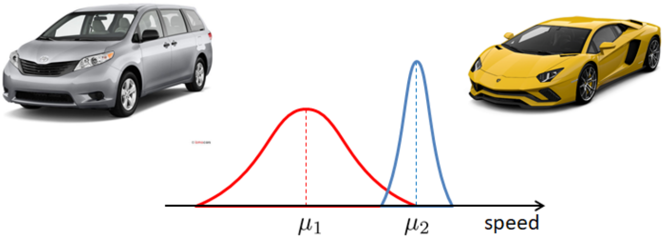
$$\begin{aligned}\mathbb{E}[X] &= \sum_y \mathbb{E}[X|Y=y]p_Y(y) \\ &= \sum_y h(y)p_Y(y) \\ &= \mathbb{E}_Y[h(Y)] = \mathbb{E}_Y\left[\mathbb{E}_{X|Y}[X|Y]\right]. \qquad \square\end{aligned}$$

Example 1

Example 1. Suppose there are two classes of cars. Let X be the speed and C be the class.

- When $C = 1$, we know that $X \sim \text{Gaussian}(\mu_1, \sigma_1)$. We know that $\mathbb{P}[C = 1] = p$.
- When $C = 2$, $X \sim \text{Gaussian}(\mu_2, \sigma_2)$.
- Also, $\mathbb{P}[C = 2] = 1 - p$.

Suppose you see a car on the freeway, what is its average speed?



Example 1

Solution. The problem has given us everything we need. In particular, we know that the conditional PDFs are:

$$f_{X|C}(x\,|\,1) =$$

$$f_{X|C}(x\,|\,2) =$$

Therefore, conditioned on C, we have two expectations:

$$\mathbb{E}[X \mid C = 1] = \qquad\qquad = \mu_1,$$

$$\mathbb{E}[X \mid C = 2] = \qquad\qquad = \mu_2.$$

The overall expectation $\mathbb{E}[X]$ is

$$\mathbb{E}[X] =$$

Example 2

Example 2. Consider a joint PMF given by the following table.

Y		$X = 0.01$	$X = 0.1$	$X = 1$	$X = 10$	$X = 100$
	10^4	0	0	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$
	10^2	$\frac{5}{12}$	$\frac{5}{18}$	$\frac{5}{36}$	0	0

Find $\mathbb{E}[X|Y = 10^2]$ and $\mathbb{E}[X|Y = 10^4]$.

Example 2

Solution. To find the conditional expectation, we first need to know the conditional PMF.

$$p_{X|Y}(x|10^2) = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} & 0 & 0 \end{bmatrix}$$
$$p_{X|Y}(x|10^4) = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}.$$

Therefore, the conditional expectations are

$$\mathbb{E}[X \mid Y = 10^2] = (\qquad)\left(\frac{1}{2}\right) + (\qquad)\left(\frac{1}{3}\right) + (\qquad)\left(\frac{1}{6}\right) = \frac{123}{600}$$
$$\mathbb{E}[X \mid Y = 10^4] = (\qquad)\left(\frac{1}{2}\right) + (\qquad)\left(\frac{1}{3}\right) + (\qquad)\left(\frac{1}{6}\right) = \frac{123}{6}.$$

From the conditional expectations we can also find $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \mathbb{E}[X \mid Y = 10^2]\,(\qquad) + \mathbb{E}[X \mid Y = 10^4]\,(\qquad)$$
$$= \left(\frac{123}{600}\right)(\qquad) + \left(\frac{123}{6}\right)(\qquad) = 3.5875.$$

Example 3

Example 3. Consider two random variables X and Y.

- The random variable X is Gaussian distributed with $X \sim \text{Gaussian}(\mu, \sigma^2)$.
- The random variable Y has a conditional distribution $Y|X \sim \text{Gaussian}(X, X^2)$.

Find $\mathbb{E}[Y]$.

Solution.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad \text{and}$$

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi x^2}} e^{-\frac{(y-x)^2}{2x^2}}.$$

Example 3

The conditional expectation of Y given X is

$$\mathbb{E}[Y|X = x] =$$

$$= \mathbb{E}[\text{Gaussian}(x, x^2)] = x.$$

So,

$$\mathbb{E}[Y] =$$

$$= \mathbb{E}[\text{Gaussian}(\mu, \sigma^2)] = \mu$$

Summary

Questions?