ECE 302: Lecture 5.6 Sum of Two Random Variables

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Sum of Two Random Variables

Given two random variables X and Y, what is the PDF of the sum, i.e., $X + Y$?

- $f_X(t) + f_Y(t)$?
- $f_X(t) \cdot f_Y(t)$?
- $f_X(x + y)$?
- ???

Outline

- Joint PDF and CDF
- Joint Expectation
- Conditional Distribution
- Conditional Expectation
- Sum of Two Random Variables
- Random Vectors
- High-dimensional Gaussians and Transformation
- Principal Component Analysis

Today's lecture

- Intuition
- Basic principle

Intuition

- X = uniform in $0, 1, 2, 3$
- Y = uniform in $0, 1, 2, 3$
- $Z = X + Y$

Find PMF of Z:

$$
\begin{aligned}
p_Z(0) &= \mathbb{P}[X + Y = 0] \\
&= \mathbb{P}[(X, Y) = (0, 0)] \\
&= p_X(0)p_Y(0) = \frac{1}{16} \\
p_Z(1) &= \mathbb{P}[X + Y = 1] \\
&= \mathbb{P}[(X, Y) = (0, 1) \cup (0, 1)] \\
&= p_X(0)p_Y(1) + p_X(1)p_Y(0) = \frac{2}{16}.
\end{aligned}
$$

Intuition

$Z = X + Y$	Cases, written in terms of (X, Y)	Probability
0	(0,0)	$1/16$
1	(0,1), (1,0)	$2/16$
2	(1,1), (2,0), (0,2)	$3/16$
3	(3,0), (2,1), (1,2), (0,3)	$4/16$
4	(3,1), (2,2), (1,3)	$3/16$
5	(3,2), (2,3)	$2/16$
6	(3,3)	$1/16$

The shape of $X + Y$

Main result

Theorem

Let X and Y be two independent random variables with PDFs $f_X(x)$ and $f_Y(y)$ respectively. Let $Z = X + Y$. The PDF of Z is given by

$$f_Z(z) = (f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy, \tag{1}$$

where "$*$" denotes the convolution.

The PDF of $X + Y$ is the convolution $f_X * f_Y$.

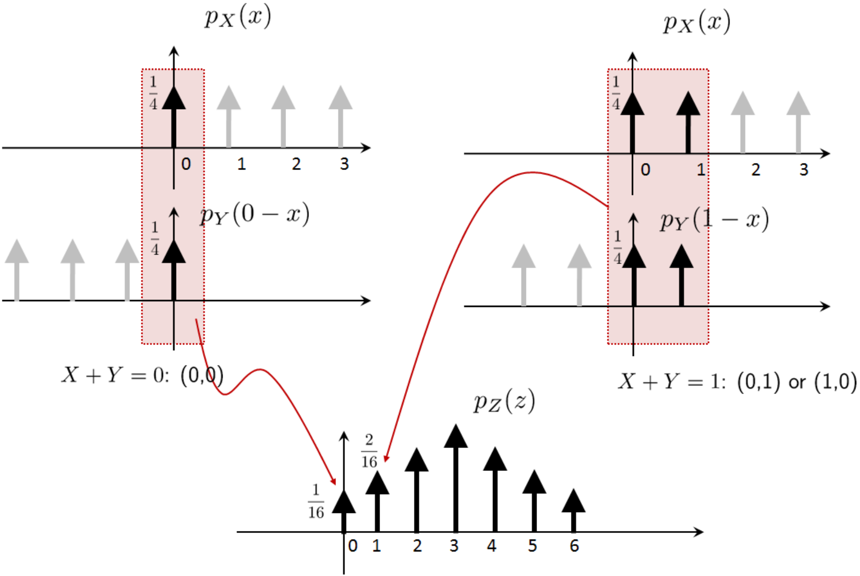


Figure: When summing two random variables X and Y, we are effectively taking the convolution of the two respective PMF / PDFs.

Proof

Always start from CDF:

$$\begin{aligned} F_Z(z) &= \mathbb{P}[Z \leq z] \\ &= \mathbb{P}[X + Y \leq z]. \end{aligned}$$

Now need to integrate:

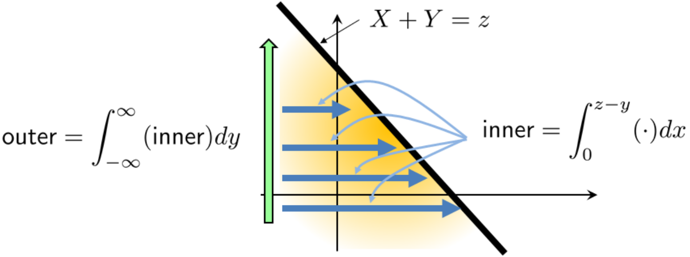


Figure: The shaded region highlights the set $X + Y \leq Z$.

Proof

$$
\begin{aligned}
\mathbb{P}[X + Y \le z] = &= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{X,Y}(x,y)dxdy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x) f_Y(y) dxdy, \quad \text{(independence)}
\end{aligned}
$$

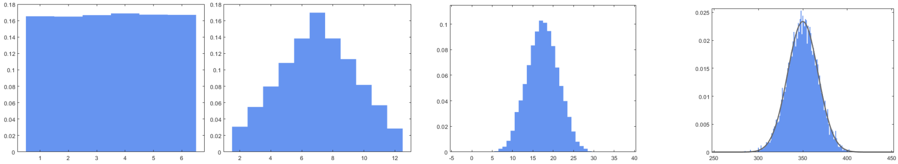
Finally, by fundamental theorem of calculus,

$$
\begin{aligned}
f_Z(z) &= \frac{d}{dz} F_Z(z) \\
&= \frac{d}{dz} \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x) f_Y(y) dxdy \\
&= \int_{-\infty}^{\infty} \left(\frac{d}{dz} \int_{-\infty}^{z-y} f_X(x) f_Y(y) dx \right) dy \\
&= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy = (f_X * f_Y)(z),
\end{aligned}
$$

Summary

How is convolution related to random variables?

- If you sum X and Y, the resulting PDF is the convolution of f_X and f_Y
- E.g., Convolving two uniform random variables give you a triangle PDF.



(a) X_1 (b) $X_1 + X_2$ (c) $X_1 + \ldots + X_5$ (d) $X_1 + \ldots + X_{100}$

Questions?