ECE 302: Lecture 5.7 Examples of $X + Y$

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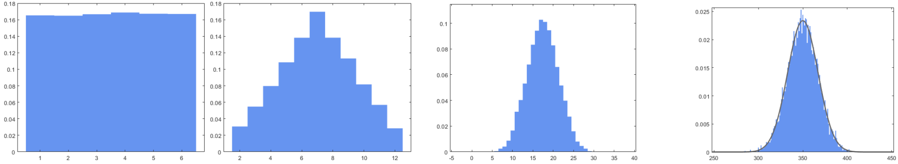
PDF of $X + Y$

What is the PDF of $X + Y$?

- If you sum X and Y, the resulting PDF is the convolution of f_X and f_Y
- E.g., Convolving two uniform random variables give you a triangle PDF.



(a) X_1 (b) $X_1 + X_2$ (c) $X_1 + \ldots + X_5$ (d) $X_1 + \ldots + X_{100}$

Outline

- Joint PDF and CDF
- Joint Expectation
- Conditional Distribution
- Conditional Expectation
- Sum of Two Random Variables
- Random Vectors
- High-dimensional Gaussians and Transformation
- Principal Component Analysis

Today's lecture

- Examples!
- Examples!!
- Examples!!!

Example 1

Example. Let X and Y be independent, and let

$$f_X(x) = \begin{cases} xe^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad f_Y(y) = \begin{cases} ye^{-y}, & y \geq 0 \\ 0, & y < 0 \end{cases}.$$

Find the PDF of $Z = X + Y$.

Example 1

Solution. Using the results derived above, we see that

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy = \int_{-\infty}^{z} f_X(z-y) f_Y(y) dy,$$

where the upper limit z came from the fact that $x \geq 0$. Therefore, since $Z = X + Y$, we must have $Z - Y = X \geq 0$ and so $Z \geq Y$. This can be visualized in the figure above. Substituting the PDFs into the integration yields

$$f_Z(z) = \int_{0}^{z} (z-y) e^{-(z-y)} y e^{-y} dy = \frac{z^3}{6} e^{-z}, \quad z \geq 0.$$

For $z < 0$, $f_Z(z) = 0$.

Example 2

Example. Let X and Y be two independent random variables such that

$$f_X(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 1, & \text{if } 0 \leq y \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$

Let $Z = XY$. Find $f_Z(z)$.

Example 2

Solution. The CDF of Z can be evaluated as

$$F_Z(z) = \mathbb{P}[Z \leq z] = \mathbb{P}[XY \leq z] = \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{z}{y}} f_X(x) f_Y(y) dx dy.$$

Taking the derivative yields

$$\begin{aligned} f_Z(z) = \frac{d}{dz} F_Z(z) &= \frac{d}{dz} \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{z}{y}} f_X(x) f_Y(y) dx dy \\ &\overset{(a)}{=} \int_{-\infty}^{\infty} \frac{1}{y} f_X(\frac{z}{y}) f_Y(y) dy, \end{aligned}$$

where (a) holds by the fundamental theorem of calculus.

Example 2

The upper and lower limit of this integration can be determined by noting that

$$0 \leq \frac{z}{y} = x \leq 1,$$

which implies that $z \leq y$. Since $y \leq 1$, we have that $z \leq y \leq 1$. Therefore, the PDF is

$$\begin{aligned} f_Z(z) &= \int_z^1 \frac{1}{y} f_X\left(\frac{z}{y}\right) f_Y(y) dy \\ &= \int_z^1 \frac{2z}{y^2} dy = 2(1-z), \quad z \geq 0. \end{aligned}$$

For $z < 0$, $f_Z(z) = 0$.

Example 3

Theorem (Sum of two Gaussians)

Let $X_1 \sim Gauss(\mu_1, \sigma_1^2)$ *and* $X_2 \sim Gauss(\mu_2, \sigma_2^2)$, *then*

$$X_1 + X_2 \sim Gaussian(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2). \tag{1}$$

Proof. Let us apply the convolution principle.

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(t) f_Y(z-t) dt \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu_1)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-t-\mu_2)^2}{2\sigma^2}} dt \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu_1)^2+(z-t-\mu_2)^2}{2\sigma^2}} dt.
\end{aligned}$$

We can now do a completing square:

$$
\begin{aligned}
&(t-\mu_1)^2 + (z-t-\mu_2)^2 \\
&= [t^2 - 2\mu_1 t + \mu_1^2] + [t^2 + 2t(\mu_2 - z) + (\mu_2 - z)^2] \\
&= 2t^2 - 2t(\mu_1 - \mu_2 + z) + \mu_1^2 + (\mu_2 - z)^2 \\
&= 2\left[t^2 - 2t \cdot \frac{\mu_1 - \mu_2 + z}{2}\right] + \mu_1^2 + (\mu_2 - z)^2 \\
&= 2\left[t - \frac{\mu_1 - \mu_2 + z}{2}\right]^2 - 2\left[\frac{\mu_1 - \mu_2 + z}{2}\right]^2 + \mu_1^2 + (\mu_2 - z)^2.
\end{aligned}
$$

The last term can be simplified as

$$
\begin{aligned}
&-2\left[\frac{\mu_1 - \mu_2 + z}{2}\right]^2 + \mu_1^2 + (\mu_2 - z)^2 \\
&= -\frac{\mu_1^2 - 2\mu_1(\mu_2 - z) + (\mu_2 - z)^2}{2} + \mu_1^2 + (\mu_2 - z)^2 \\
&= \frac{\mu_1^2 + 2\mu_1(\mu_2 - z) + (\mu_2 - z)^2}{2} = \frac{(\mu_1 + \mu_2 - z)^2}{2}.
\end{aligned}
$$

Substituting these into the integral, we can show that

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{2\left[t-\frac{\mu_1-\mu_2+z}{2}\right]^2+\frac{(\mu_1+\mu_2-z)^2}{2}}{2\sigma^2}} dt$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\mu_1+\mu_2-z)^2}{2(2\sigma^2)}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left[t-\frac{\mu_1-\mu_2+z}{2}\right]^2}{\sigma^2}} dt}_{=\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{\sqrt{2\pi(2\sigma)^2}} e^{-\frac{(\mu_1+\mu_2-z)^2}{2(2\sigma^2)}}.$$

Therefore, we have shown that the resulting distribution is a Gaussian with mean $\mu_1 + \mu_2$, and variance $2\sigma^2$.

Summary

Steps to do sum of two variables:

- Find $F_Z(z) = \mathbb{P}[Z \leq z]$.
- Determine the upper and lower limit for the integration.
- Find $f_Z(z) = \frac{d}{dz} F_Z(z)$.

X_1	X_2	Sum $X_1 + X_2$
Bernoulli(p)	Bernoulli(p)	Binomial($2, p$)
Binomial(n, p)	Binomial(m, p)	Binomial($m + n, p$)
Poisson(λ_1)	Poisson(λ_2)	Poisson($\lambda_1 + \lambda_2$)
Exponential(λ)	Exponential(λ)	Erlang($2, \lambda$)
Gaussian(μ_1, σ_1^2)	Gaussian(μ_2, σ_2^2)	Gaussian($\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2$)

Questions?