ECE 302: Lecture 5.9 Multi-dimensional Gaussian

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Random vectors

Random vector:

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}, \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}.$$

Mean vector:

$$\boldsymbol{\mu} \overset{\text{def}}{=} \mathbb{E}[\boldsymbol{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \\ \vdots \\ \mathbb{E}[X_N] \end{bmatrix} \tag{1}$$

Covariance:

$$\boldsymbol{\Sigma} \overset{\text{def}}{=} \mathrm{Cov}(\boldsymbol{X}) = \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}(X_1, X_2) & \dots & \mathrm{Cov}(X_1, X_N) \\ \mathrm{Cov}[X_2, X_1] & \mathrm{Var}[X_2] & \dots & \mathrm{Cov}(X_2, X_N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(X_N, X_1) & \mathrm{Cov}(X_N, X_2) & \dots & \mathrm{Var}[X_N] \end{bmatrix} \tag{2}$$

Multi-dimensional Gaussian

Definition

A d-dimensional **joint Gaussian** has a PDF

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^d |\boldsymbol{\Sigma}|}} \exp\left\{ -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) \right\}, \tag{3}$$

where d denotes the dimensionality of the vector $\boldsymbol{x}$.

What happens if X_i's are independent?

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_d^2 \end{bmatrix},$$

Then, the exponential becomes

$$(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) = \begin{bmatrix} x_1-\mu_1 \\ \vdots \\ x_d-\mu_d \end{bmatrix}^T \begin{bmatrix} \sigma_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_d^2 \end{bmatrix}^{-1} \begin{bmatrix} x_1-\mu_1 \\ \vdots \\ x_d-\mu_d \end{bmatrix}$$

$$= \begin{bmatrix} x_1-\mu_1 \\ \vdots \\ x_d-\mu_d \end{bmatrix}^T \begin{bmatrix} \frac{x_1-\mu_1}{\sigma_1^2} \\ \vdots \\ \frac{x_d-\mu_d}{\sigma_d^2} \end{bmatrix} = \sum_{i=1}^{d} \frac{(x_i-\mu_i)^2}{\sigma_i^2}.$$

What happens if X_i's are independent?

The determinant becomes

$$|\boldsymbol{\Sigma}| = \left| \begin{bmatrix} \sigma_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_d^2 \end{bmatrix} \right| = \prod_{i=1}^{d} \sigma_i^2.$$

So the PDF is

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{(2\pi)\sigma_i^2}} \exp\left\{ -\frac{(x-\mu_i)^2}{2\sigma_i^2} \right\} \tag{4}$$

Generating multi-dimensional Gaussian

```
% MATLAB
mu    = [0 0];
Sigma = [.25 .3; .3 1];
x     = mvnrnd(mu,Sigma,1000);
```

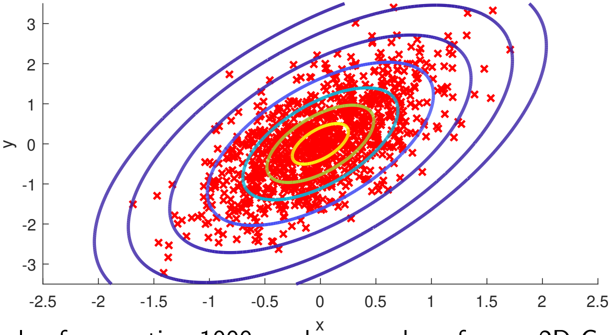


Figure: Example of generating 1000 random numbers from a 2D Gaussian and plotting the contour.

Generating multi-dimensional Gaussian

```
% MATLAB
x1 = -2.5:.01:2.5;
x2 = -3.5:.01:3.5;
[X1,X2] = meshgrid(x1,x2);
F = mvnpdf([X1(:) X2(:)],mu,Sigma);
F = reshape(F,length(x2),length(x1));
figure(1);
scatter(x(:,1),x(:,2),'rx', 'LineWidth', 1.5); hold on;
contour(x1,x2,F,[.001 .01 .05:.1:.95 .99 .999], ...
'LineWidth', 2);
xlabel('x'); ylabel('y');
set(gcf, 'Position', [100, 100, 600, 300]);
```

Eigenvalues and eigenvectors

How do we determine the shape of a Gaussian?
By eigenvalue decomposition.

Definition

Given a square matrix $\boldsymbol{A} \in \mathbb{R}^{N \times N}$, the vector $\boldsymbol{u} \in \mathbb{R}^N$ (with $\boldsymbol{u} \neq \mathbf{0}$) is called the **eigenvector** of $\boldsymbol{A}$ if

$$\boldsymbol{A}\boldsymbol{u} = \lambda \boldsymbol{u}, \tag{5}$$

for some $\lambda \in \mathbb{R}$. The scalar λ is called the **eigenvalue** associated with $\boldsymbol{u}$.

- There exists $\boldsymbol{u} \neq 0$ such that $\boldsymbol{A}\boldsymbol{u} = \lambda \boldsymbol{u}$;
- There exists $\boldsymbol{u} \neq 0$ such that $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{u} = \mathbf{0}$;
- $(\boldsymbol{A} - \lambda \boldsymbol{I})$ is not invertible;
- $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$;

Eigen-decomposition

Theorem

If $\boldsymbol{A}$ is symmetric, then all the eigenvalues are real, and there exists $\boldsymbol{U}$ such that $\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}$ and $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T$:

$$\underbrace{\begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_N \\ | & | & & | \end{bmatrix}}_{\boldsymbol{A}} = \underbrace{\begin{bmatrix} | & | & & | \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \ldots & \boldsymbol{u}_N \\ | & | & & | \end{bmatrix}}_{\boldsymbol{U}} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix}}_{\boldsymbol{\Lambda}} \underbrace{\begin{bmatrix} — & \boldsymbol{u}_1^T & — \\ — & \boldsymbol{u}_2^T & — \\ & \vdots & \\ — & \boldsymbol{u}_N^T & — \end{bmatrix}}_{\boldsymbol{U}^T}.$$

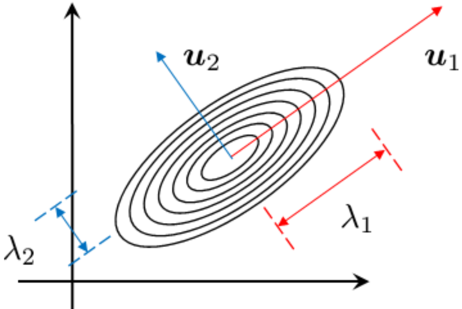
Eigen-decomposition on MATLAB

```
% MATLAB Code:
A = randn(100,100);
A = (A + A')/2;       % symmetrize because A is not symmetric
[U,S] = eig(A);       % eigen-decomposition
s = diag(S);          % extract eigen-value
```

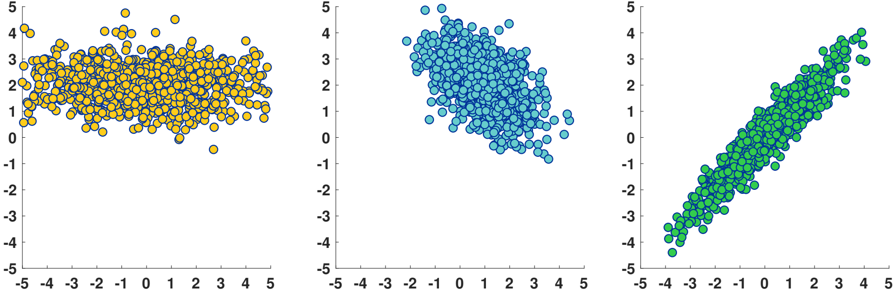
Interpreting the eigen-vectors

$$\boldsymbol{\Sigma} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T$$

$$= \begin{bmatrix} | & | & & | \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \dots & \boldsymbol{u}_d \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \lambda_d \end{bmatrix} \begin{bmatrix} - & \boldsymbol{u}_1^T & - \\ - & \boldsymbol{u}_2^T & - \\ & \vdots & \\ - & \boldsymbol{u}_d^T & - \end{bmatrix}. \tag{6}$$



Examples



$$(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1.9 \\ 1.9 & 2 \end{bmatrix}$$

Examples

Example 1:

$$\text{eig}\left(\begin{bmatrix} 5 & 0 \\ 0 & 0.5 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0.5 & 0 \\ 0 & 5 \end{bmatrix}$$

Example 2:

$$\text{eig}\left(\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}\right) = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}, \begin{bmatrix} 0.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

Example 3:

$$\text{eig}\left(\begin{bmatrix} 2 & 1.9 \\ 1.9 & 2 \end{bmatrix}\right) = \begin{bmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}, \begin{bmatrix} 0.1 & 0 \\ 0 & 3.9 \end{bmatrix}$$

Will you get negative eigenvalues?

You will **never** get negative eigenvalues for a covariance matrix.

Theorem

The covariance matrix $\mathrm{Cov}(\boldsymbol{X}) = \boldsymbol{\Sigma}$ *is symmetric positive semi-definite, i.e.,*

$$\boldsymbol{\Sigma}^T = \boldsymbol{\Sigma}, \quad \textit{and} \quad \boldsymbol{v}^T \boldsymbol{\Sigma} \boldsymbol{v} \geq 0, \quad \forall \boldsymbol{v} \in \mathbb{R}^d.$$

Proof. See ebook.

Summary

Definition

A d-dimensional **joint Gaussian** has a PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d |\mathbf{\Sigma}|}} \exp\left\{ -\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\}, \tag{7}$$

where d denotes the dimensionality of the vector $\mathbf{x}$.

- Shape and orientation is determined by the covariance
- Use of eigen-decomposition
- Covariance is always positive semi-definite

Questions?