ECE 302: Lecture 5.10 Gaussian Whitening

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Multi-dimensional Gaussian

Definition

A d-dimensional **joint Gaussian** has a PDF

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^d|\boldsymbol{\Sigma}|}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}, \tag{1}$$

where d denotes the dimensionality of the vector $\boldsymbol{x}$.

Question

From Gaussian$(0, \boldsymbol{I})$ **to Gaussian**$(\mu, \boldsymbol{\Sigma})$.

If we are given $\boldsymbol{X} \sim \text{Gaussian}(\boldsymbol{0}, \boldsymbol{I})$,
how do we generate $\boldsymbol{Y} \sim \text{Gaussian}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ from $\boldsymbol{X}$?

From Gaussian$(\mu, \boldsymbol{\Sigma})$ **to Gaussian**$(0, \boldsymbol{I})$.

If we are given $\boldsymbol{Y} \sim \text{Gaussian}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$,
how do we generate $\boldsymbol{X} \sim \text{Gaussian}(\boldsymbol{0}, \boldsymbol{I})$ from $\boldsymbol{Y}$?

First result

Theorem

Let $\boldsymbol{X}$ be a zero-mean unit-variance Gaussian $\boldsymbol{X} \sim \text{Gaussian}(\boldsymbol{0}, \boldsymbol{I})$. Consider a mean vector $\boldsymbol{\mu}$ and a covariance matrix $\boldsymbol{\Sigma}$ with eigen-decomposition $\boldsymbol{\Sigma} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T$. If

$$\boldsymbol{Y} = \boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{X} + \boldsymbol{\mu}, \tag{2}$$

where $\boldsymbol{\Sigma}^{\frac{1}{2}} = \boldsymbol{U}\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{U}^T$, then $\boldsymbol{Y} \sim \text{Gaussian}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

- Step 1: Generate samples $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ which are distributed according to Gaussian$(\boldsymbol{0}, \boldsymbol{I})$.
- Step 2: Define $\boldsymbol{y}_n$ where

$$\boldsymbol{y}_n = \boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{x}_n + \boldsymbol{\mu}. \tag{3}$$

Main idea

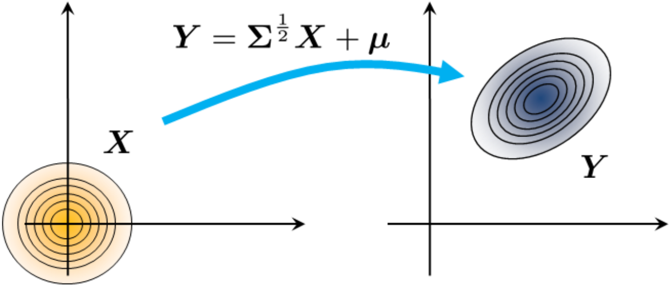


Figure: Generating an arbitrary Gaussian from Gaussian$(\mathbf{0}, \boldsymbol{I})$.

Proof

The idea is to define a transformation

$$\boldsymbol{Y} = \boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{X} + \boldsymbol{\mu}, \tag{4}$$

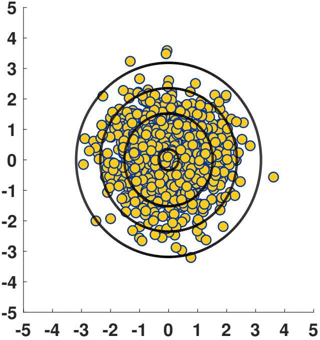
where $\boldsymbol{\Sigma}^{\frac{1}{2}} = \boldsymbol{U}\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{U}^T$. Then, the mean of $\boldsymbol{Y}$ is

$$\mathbb{E}[\boldsymbol{Y}] = \mathbb{E}[\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{X} + \boldsymbol{\mu}] = \boldsymbol{\Sigma}^{\frac{1}{2}}\mathbb{E}[\boldsymbol{X}] + \boldsymbol{\mu} = \boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{0} + \boldsymbol{\mu} = \boldsymbol{\mu}.$$

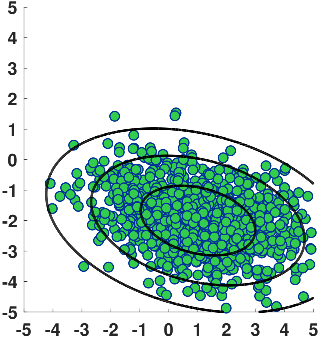
and the covariance matrix is

$$\begin{aligned}
\mathbb{E}[(\boldsymbol{Y} - \boldsymbol{\mu})(\boldsymbol{Y} - \boldsymbol{\mu})^T] &= \mathbb{E}[(\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{X} + \boldsymbol{\mu} - \boldsymbol{\mu})(\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{X} + \boldsymbol{\mu} - \boldsymbol{\mu})^T] \\
&= \mathbb{E}[(\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{X})(\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{X})^T] = \boldsymbol{\Sigma}^{\frac{1}{2}}\mathbb{E}[\boldsymbol{X}\boldsymbol{X}^T]\boldsymbol{\Sigma}^{\frac{1}{2}} \\
&= \boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{I}\boldsymbol{\Sigma}^{\frac{1}{2}} = \boldsymbol{\Sigma}.
\end{aligned}$$

Example



(a) Before



(b) After

Figure: Generating arbitrary Gaussian random variables from Gaussian($\mathbf{0}, \boldsymbol{I}$).

Second result

Theorem

Let $\boldsymbol{Y}$ be a Gaussian $\boldsymbol{Y} \sim \text{Gaussian}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. If

$$\boldsymbol{X} = \boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{Y} - \boldsymbol{\mu}). \tag{5}$$

then $\boldsymbol{X} \sim \text{Gaussian}(\mathbf{0}, \boldsymbol{I})$.

- Step 1: Given samples $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_N$ which are distributed according to Gaussian$(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.
- Step 2: Define x_n where

$$\boldsymbol{x}_n = \boldsymbol{\Sigma}^{\frac{1}{2}}(\boldsymbol{y}_n - \boldsymbol{\mu}). \tag{6}$$

Main idea

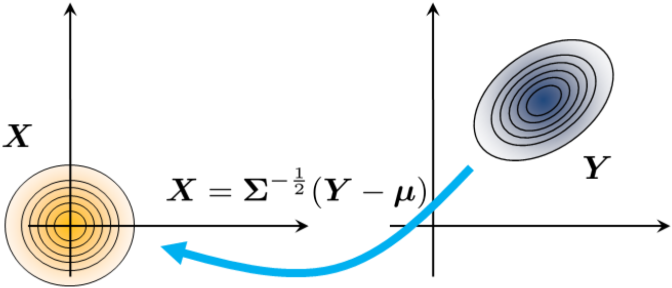


Figure: Converting an arbitrary Gaussian back to Gaussian$(\mathbf{0}, \boldsymbol{I})$.

Proof

Suppose that we have $\boldsymbol{Y} \sim \text{Gaussian}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we can define

$$\boldsymbol{X} = \boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{Y} - \boldsymbol{\mu}). \tag{7}$$

Then,

$$\mathbb{E}[\boldsymbol{X}] = \mathbb{E}[\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{Y} - \boldsymbol{\mu})] = \boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbb{E}[\boldsymbol{Y}] - \boldsymbol{\mu}) = \mathbf{0}.$$

The covariance is

$$\begin{aligned}
\mathrm{Cov}(\boldsymbol{X}) = \mathbb{E}[(\boldsymbol{X} - \boldsymbol{\mu}_X)(\boldsymbol{X} - \boldsymbol{\mu}_X)^T] &= \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^T] \\
&= \mathbb{E}\left[\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{Y} - \boldsymbol{\mu})(\boldsymbol{Y} - \boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-\frac{T}{2}}\right] \\
&= \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbb{E}\left[(\boldsymbol{Y} - \boldsymbol{\mu})(\boldsymbol{Y} - \boldsymbol{\mu})^T\right]\boldsymbol{\Sigma}^{-\frac{T}{2}} \\
&= \boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-\frac{1}{2}} = \boldsymbol{I}.
\end{aligned}$$

Summary

From $\boldsymbol{X} \sim \text{Gaussian}(\mathbf{0}, \boldsymbol{I})$ to $\boldsymbol{Y} \sim \text{Gaussian}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

- Step 1: Generate samples $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ which are distributed according to $\text{Gaussian}(\mathbf{0}, \boldsymbol{I})$.
- Step 2: Define $\boldsymbol{y}_n$ where $\boldsymbol{y}_n = \boldsymbol{\Sigma}^{\frac{1}{2}} \boldsymbol{x}_n + \boldsymbol{\mu}$

From $\boldsymbol{Y} \sim \text{Gaussian}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ to $\boldsymbol{X} \sim \text{Gaussian}(\mathbf{0}, \boldsymbol{I})$

- Step 1: Given samples $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_N$ which are distributed according to $\text{Gaussian}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.
- Step 2: Define x_n where $\boldsymbol{x}_n = \boldsymbol{\Sigma}^{\frac{1}{2}} (\boldsymbol{y}_n - \boldsymbol{\mu})$.

Questions?