ECE 302: Lecture 6.1 Moment Generating Function

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Moment Generating Function

Definition

For any random variable X, the **moment generating function** (MGF) $M_X(s)$ is

$$M_X(s) = \mathbb{E}\left[e^{sX}\right]. \tag{1}$$

Discrete:

$$M_X(s) = \sum_{x \in \Omega} e^{sx} p_X(x) \tag{2}$$

Continuous:

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x) dx \tag{3}$$

Interpretation: Laplace transform:

$$\mathcal{L}[f](s) = \int_{-\infty}^{\infty} f(t) e^{st} dt.$$

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Example 1

Example. Consider a random variable X with three state $0, 1, 2$ with probability masses $\frac{2}{6}, \frac{3}{6}, \frac{1}{6}$ respectively. Find MGF.

Solution.

$$M_X(s) =$$

$$= \frac{1}{3} + \frac{e^s}{2} + \frac{e^{2s}}{6}.$$

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Example 2

Example. Find the MGF for a Poisson random variable.

Solution. The MGF is

$$M_X(s) =$$

$$= e^{\lambda e^s} e^{-\lambda}.$$

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Example 3

Example. Find the MGF for an exponential random variable.

Solution. The MGF is

$$M_X(s) =$$

$$= \frac{\lambda}{\lambda - s}, \qquad \text{if } \lambda > s.$$

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Getting Moments from MGF

Theorem

The MGF has the property that

- $M_X(0) = 1$,
- $\frac{d^k}{ds^k} M_X(s)|_{s=0} = \mathbb{E}[X^k]$, *for any positive integer* k.

Proof. The first property can be proved by noting that

$$M_X(0) = \mathbb{E}[e^{0X}] = \mathbb{E}[1] = 1.$$

The second property holds because

$$\frac{d^k}{ds^k} M_X(s) = \int_{-\infty}^{\infty} \frac{d^k}{ds^k} e^{sx} f_X(x) dx = \int_{-\infty}^{\infty} x^k e^{sx} f_X(x) dx.$$

Setting $s = 0$ yields

$$\frac{d^k}{ds^k} M_X(s)|_{s=0} = \int_{-\infty}^{\infty} x^k f_X(x) dx = \mathbb{E}[X^k].$$

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Moment Generating Functions

Distribution	PMF/ PDF	$\mathbb{E}[X]$	$\mathrm{Var}[X]$	$M_X(s)$
Bernoulli	$\mathbb{P}[X=1]=p$	p	$p(1-p)$	$M_X(s)=1-p+pe^s$
Binomial	$p_X(k)=\binom{n}{k}p^k(1-p)^{n-k}$	np	$np(1-p)$	$M_X(s)=(1-p+pe^s)^n$
Geometric	$p_X(k)=p(1-p)^{k-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$M_X(s)=\frac{pe^s}{1-(1-p)e^s}$
Poisson	$p_X(k)=\frac{\lambda^k e^{-\lambda}}{k!}$	λ	λ	$M_X(s)=e^{\lambda(e^s-1)}$
Gaussian	$f_X(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2	$M_X(s)=e^{\mu s+\frac{\sigma^2 s^2}{2}}$
Exponential	$f_X(x)=\lambda\exp\{-\lambda x\}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$M_X(s)=\frac{\lambda}{\lambda-s}$
Uniform	$f_X(x)=\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$M_X(s)=\frac{e^{sb}-e^{sa}}{s(b-a)}$

Table: Moment generating functions of common random variables.

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Independent random variables

Theorem

Let X and Y be independent random variables. Let $Z = X + Y$. Then,

$$M_Z(s) = M_X(s)M_Y(s). \tag{4}$$

Proof. By definition of MGF, we have that

$$M_Z(s) = \mathbb{E}\left[e^{s(X+Y)}\right] \overset{(a)}{=} \mathbb{E}\left[e^{sX}\right]\mathbb{E}\left[e^{sY}\right] = M_X(s)M_Y(s),$$

where (a) holds because X and Y are independent. □

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Many Independent Random Variables

Corollary

- *independent random variables* $X_1, \ldots, X_N$
- $Z = \sum_{n=1}^{N} X_n$

Then, the MGF of Z *is*

$$M_Z(s) = \prod_{n=1}^{N} M_{X_n}(s).$$

If $X_1, \ldots, X_N$ *are i.i.d., then the MGF is*

$$M_Z(s) = \left(M_{X_1}(s)\right)^N.$$

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Example 4

Theorem (Sum of Bernoulli = Binomial)

- $X_1, \ldots, X_N$ *be a sequence of i.i.d. Bernoulli random variables with parameter* p
- $Z = X_1 + \ldots + X_N$

Then Z *is a binomial random variable with parameters* (N, p).

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Proof

MGF of Z is

$$M_Z(s) = \mathbb{E}[e^{s(X_1+\ldots+X_N)}] = \prod_{n=1}^{N} \mathbb{E}[e^{sX_n}]$$

$$= \prod_{n=1}^{N} \left(pe^{s1} + (1-p)e^{s0}\right) = \left(pe^{s} + (1-p)\right)^N.$$

MGF of a binomial random variable: If $Z \sim \mathrm{Binomial}(N, p)$, then

$$M_Z(s) = \mathbb{E}[e^{sZ}] = \sum_{n=0}^{N} e^{sk} \binom{N}{k} p^k (1-p)^{N-k}$$

$$= \sum_{n=0}^{N} \binom{N}{k} (pe^s)^k (1-p)^{N-k} = \left(pe^s + (1-p)\right)^N,$$

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Example 5

Theorem (Sum of Gaussian = Gaussian)

Let $X_1, \ldots, X_N$ be a sequence of Gaussian random variables with parameters $(\mu_1, \sigma_1), \ldots (\mu_N, \sigma_N)$. Let $Z = X_1 + \ldots + X_N$ be the sum. Then, Z is a Gaussian random variable:

$$Z = Gaussian\left(\sum_{n=1}^{N} \mu_n, \sum_{n=1}^{N} \sigma_n^2\right). \tag{5}$$

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Proof

Proof. We skip the proof of the MGF of a Gaussian. It can be shown that

$$M_X(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}. \tag{6}$$

When we have a sequence of Gaussian random variables, then

$$\begin{aligned}
M_Z(s) &= \mathbb{E}[e^{s(X_1 + \ldots + X_N)}] \\
&= M_{X_1}(s) \ldots M_{X_N}(s) \\
&= \left(e^{\mu_1 s + \frac{\sigma_1^2 s^2}{2}} \right) \cdots \left(e^{\mu_N s + \frac{\sigma_N^2 s^2}{2}} \right) \\
&= \exp\left\{ \left(\sum_{n=1}^{N} \mu_n \right) s + \left(\sum_{n=1}^{N} \sigma_n^2 \right) \frac{s^2}{2} \right\}.
\end{aligned}$$

Therefore, the resulting random variable Z is also a Gaussian. The mean and variance are $\sum_{n=1}^{N} \mu_n$ and $\sum_{n=1}^{N} \sigma_n^2$, respectively. □

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Summary

What is a moment generating function?

- $\mathbb{E}[e^{sX}]$
- Can generate moments
- Useful for sum of random variables

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Questions?

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