

# ECE 302: Lecture 6.2 Characteristic Function

Prof Stanley Chan

School of Electrical and Computer Engineering  
Purdue University



# Characteristic Function

## Definition (This course)

The **characteristic function** of a random variable  $X$  is

$$\Phi_X(j\omega) = \mathbb{E}[e^{-j\omega X}]. \quad (1)$$

## Definition (Common definition)

The **characteristic function** of a random variable  $X$  is

$$\Phi_X(j\omega) = \mathbb{E}[e^{j\omega X}]. \quad (2)$$

Recall MGF is

$$M_X(s) = \mathbb{E}[e^{sX}]. \quad (3)$$

So,  $\Phi_X(j\omega)$  is MGF evaluated at  $s = -j\omega$ .

# What is characteristic function then?

## Definition (This course)

The **characteristic function** of a random variable  $X$  is

$$\Phi_X(j\omega) = \mathbb{E}[e^{-j\omega X}]. \quad (4)$$

Characteristic function is the Fourier transform of  $f_X(x)$ :

$$\Phi_X(j\omega) = \mathbb{E}[e^{-j\omega X}] = \int_{-\infty}^{\infty} e^{-j\omega x} f_X(x) dx. \quad (5)$$

## Fourier transform Table

$f(t) \longleftrightarrow F(\omega)$	$f(t) \longleftrightarrow F(\omega)$
1. $e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}, a > 0$	10. $\text{sinc}^2\left(\frac{Wt}{2}\right) \longleftrightarrow \frac{2\pi}{W} \Delta\left(\frac{\omega}{2W}\right)$
2. $e^{at}u(-t) \longleftrightarrow \frac{1}{a-j\omega}, a > 0$	11. $e^{-at} \sin(\omega_0 t)u(t) \longleftrightarrow \frac{\omega_0}{(a+j\omega)^2 + \omega_0^2}$
3. $e^{-a t } \longleftrightarrow \frac{2a}{a^2 + \omega^2}, a > 0$	12. $e^{-at} \cos(\omega_0 t)u(t) \longleftrightarrow \frac{a+j\omega}{(a+j\omega)^2 + \omega_0^2}$
4. $\frac{a^2}{a^2 + t^2} \longleftrightarrow \pi a e^{-a \omega }, a > 0$	13. $e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \sqrt{2\pi}\sigma e^{-\frac{\sigma^2\omega^2}{2}}$
5. $te^{-at}u(t) \longleftrightarrow \frac{1}{(a+j\omega)^2}, a > 0$	14. $\delta(t) \longleftrightarrow 1$
6. $t^n e^{-at}u(t) \longleftrightarrow \frac{n!}{(a+j\omega)^{n+1}}, a > 0$	15. $1 \longleftrightarrow 2\pi\delta(\omega)$
7. $\text{rect}\left(\frac{t}{\tau}\right) \longleftrightarrow \tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$	16. $\delta(t - t_0) \longleftrightarrow e^{-j\omega t_0}$
8. $\text{sinc}(Wt) \longleftrightarrow \frac{\pi}{W} \text{rect}\left(\frac{\omega}{2W}\right)$	17. $e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - \omega_0)$
9. $\Delta\left(\frac{t}{\tau}\right) \longleftrightarrow \frac{\tau}{2} \text{sinc}^2\left(\frac{\omega\tau}{4}\right)$	18. $f(t)e^{j\omega_0 t} \longleftrightarrow F(\omega - \omega_0)$

## Example 1

**Example.** Let  $X$  be a random variable with PDF  $f_X(x) = \lambda e^{-\lambda x}$  for  $x \geq 0$ . Find the characteristic function.

**Solution.** The Fourier transform pair is

$$\lambda e^{-\lambda x} \longrightarrow \lambda \cdot \mathcal{F}\left\{e^{-\lambda x}\right\} = \lambda \cdot \frac{1}{\lambda + j\omega}.$$

Therefore, the characteristic function is  $\Phi_X(j\omega) = \frac{\lambda}{\lambda + j\omega}$ .

## Example 2

**Example.** Let  $X$  and  $Y$  be independent, and let

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y \geq 0 \\ 0, & y < 0 \end{cases}.$$

Find the PDF of  $Z = X + Y$ .

## Example 2

**Solution.** The characteristic function of  $X$  and  $Y$  can be found from the Fourier table:

$$\Phi_X(j\omega) = \frac{\lambda}{\lambda + j\omega}, \quad \text{and} \quad \Phi_Y(j\omega) = \frac{\lambda}{\lambda + j\omega}.$$

Therefore, the characteristic function of  $Z$  is

$$\Phi_Z(j\omega) = \Phi_X(j\omega)\Phi_Y(j\omega) = \frac{\lambda^2}{(\lambda + j\omega)^2}.$$

By inverse Fourier transform, we have that

$$f_Z(z) = \mathcal{F}^{-1} \left\{ \frac{\lambda^2}{(\lambda + j\omega)^2} \right\} = \lambda^2 z e^{-\lambda z}, \quad z \geq 0.$$

## Example 3

**Example.** Let  $X_0, X_1, \dots$  be a sequence of independent random variables with PDF

$$f_{X_k}(x) = \frac{a_k}{\pi(a_k^2 + x^2)}, \quad a_k = \frac{1}{2^{k+1}},$$

for  $k = 0, 1, \dots$ . Find the PDF of  $Y$ , where  $Y = \sum_{k=0}^{\infty} X_k$ .



## Example 3

**Solution.** Using the Fourier transform table, we know that

$$\frac{a_k}{\pi(a_k^2 + x^2)} \xleftrightarrow{\mathcal{F}}$$

The characteristic function of  $Y$  is

$$\Phi_Y(j\omega) = \exp \left\{ -|\omega| \sum_{k=0}^{\infty} a_k \right\}.$$

Since  $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2} + \frac{1}{4} + \dots = 1$ , the characteristic function becomes  $\Phi_Y(j\omega) = e^{-|\omega|}$ . Inverse Fourier transform will give us

$$e^{-|\omega|} \xleftrightarrow{\mathcal{F}}$$

Therefore the PDF of  $Y$  is

$$f_Y(y) = \frac{1}{\pi(1 + y^2)}. \quad (6)$$

## Why $\Phi_X(j\omega)$ but not $M_X(s)$ ?

- MGF does not always exist

### Theorem

*Consider the Cauchy distribution with PDF*

$$f_X(x) = \frac{1}{\pi(x^2 + 1)}.$$

*The MGF of  $X$  is undefined, but the characteristic function is well defined.*

## Proof

Proof. It can be shown that the MGF is

$$\begin{aligned}M_X(s) &= \int_{-\infty}^{\infty} e^{sx} \frac{1}{\pi(x^2 + 1)} dx \geq \int_1^{\infty} e^{sx} \frac{1}{\pi(x^2 + 1)} dx \\ &\geq \int_1^{\infty} \frac{(sx)^3}{6\pi(x^2 + 1)} dx, \quad \text{because } e^{sx} \geq \frac{(sx)^3}{6} \\ &\geq \int_1^{\infty} \frac{(sx)^3}{6\pi(2x^2)} dx = \frac{s^3}{12\pi} \int_1^{\infty} x dx = \infty.\end{aligned}$$

Therefore, the MGF is undefined. On the other hand, by Fourier table we know that

$$\Phi_X(j\omega) = \mathcal{F} \left\{ \frac{1}{\pi(x^2 + 1)} \right\} = e^{-|\omega|}. \quad \square$$

## What is a characteristic function?

- $\mathbb{E}[e^{-j\omega X}]$
- Also okay to define as  $\mathbb{E}[e^{j\omega X}]$
- Fourier transform of PDF
- Useful for sum of random variables

**Questions?**